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Wave propagation in a simplified modelled poroelastic continuum: Fundamental solutions and a time domain boundary element formulation

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SUMMARY

In finite element formulations for poroelastic continua a representation of Biot's theory using the unknowns solid displacement and pore pressure is preferred. Such a formulation is possible either for quasi-static problems or for dynamic problems if the inertia effects of the fluid are neglected. Contrary to these formulations a boundary element method (BEM) for the general case of Biot's theory in time domain has been published (*Wave Propagation in Viscoelastic and Poroelastic Continua*: *A Boundary Element Approach*. Lecture Notes in Applied Mechanics. Springer: Berlin, Heidelberg, New York, 2001.). If the advantages of both methods are required it is common practice to couple both methods. However, for such a coupled FE/BE procedure a BEM for the simplified dynamic Biot theory as used in FEM must be developed.

Therefore, here, the fundamental solutions as well as a BE time stepping procedure is presented for the simplified dynamic theory where the inertia effects of the fluid are neglected. Further, a semi-analytical one-dimensional solution is presented to check the proposed BE formulation. Finally, wave propagation problems are studied using either the complete Biot theory as well as the simplified theory. These examples show that no significant differences occur for the selected material. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: simplified poroelasticity; Biot's theory; time domain BEM; CQM

1. INTRODUCTION

A historical review on the subject of multiphase continuum mechanics identifies two poroelastic theories which have been developed and are used nowadays, namely Biot's theory and the

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Theory of Porous Media. For more details, the reader is directed to the work of de Boer [1, 2] or to the recently published monograph [3].

Based on the work of von Terzaghi, a theoretical description of porous materials saturated by a viscous fluid was presented by Biot [4]. The dynamic extension was done in two papers, one for the low frequency range [5] and the other for the high frequency range [6]. Based on the work of Fillunger, the Theory of Porous Media has been developed. This theory is based on the axioms of continuum theories of mixtures [7, 8] extended by the concept of volume fractions by Bowen [9, 10] and others [11, 12]. Remarks on the equivalence of both theories, which model the same physical phenomenon, are found in the work of Bowen [10], Ehlers and Kubik [13], and Schanz and Diebels [14]. In all these publications, linear version of both theories are compared. Summarizing the results, it is found that for incompressible constituents both theories are identical if Biot's apparent mass density is set to zero. Further, in case of compressible constituents, the governing mathematical operator is identical but differ in the constant coefficients, i.e. the physical constants used are different. It is still an open question how this gap can be bridged.

Here, Biot's theory is used but the results can be simply transferred to the Theory of Porous Media because in the following the apparent mass density will be neglected and the equivalence of the mathematical operator ensures to have the same fundamental solutions however with different material constants.

In all above-mentioned versions of a poroelastic theory, the question arise which set of unknowns is used to formulate the set of governing differential equations. In the most general case, the vector of the solid displacement, the vector of the seepage velocity, and the pore pressure are used to derive the governing equations. Clearly, the seepage velocity can be substituted by either the relative fluid to solid displacement vector or by the fluid displacement vector itself. But, this does not change the amount of degrees of freedom (dof). In total there are seven dof in a three-dimensional (3-d) formulation and five dof in a two-dimensional (2-d) formulation.

Either from a physical point of view as well as from a numerical point of view a reduction of these dof is desirable. Usually a fluid is described by a scalar value like the pressure and a solid by a vector quantity like the displacement vector. This can also be done here resulting in a sufficient set of unknowns [15], i.e. the solid displacement vector is chosen to describe the solid skeleton and the pore pressure for the fluid. However, this requires the elimination of the seepage velocity. Because the seepage velocity is given in a differential equation with respect to time by the balance law of the fluid, i.e. by Darcy's law, its elimination is only possible in a transformed domain, e.g. Laplace or Fourier domain [5]. For modelling consolidation a quasi-static model is used, i.e. inertia effects are neglected, and, therefore, this elimination is even possible in time domain. However, the aimed application is wave propagation so such a simplification is not possible.

To avoid these difficulties in the finite element (FEM) literature on poroelastic wave propagation a simplified poroelastic model is introduced to be able to formulate and solve the governing differential equations directly in the time domain [16]. This simplification neglects only the inertia effects of the fluid but not those of the solid skeleton. In the following, this approach will be called *simple poro*. The applicability of this approach has been studied by Zienkiewicz *et al.* [17] showing that problems with low frequency accelerations can be treated well by this approach, e.g. applications in earthquake engineering.

In contrast to the FEM, for the boundary element method (BEM) no fundamental solution and, therefore, no BE formulation has been published for the simple poro model. This is due to the availability of a time domain formulation of the general poroelastic model [18]. This BE formulation is based on the Laplace domain fundamental solutions using the convolution quadrature method proposed by Lubich [19, 20]. The usage of the Laplace domain solutions avoids any difficulties with the elimination of the seepage velocity. However, also for treating wave propagation problems in a non-linear poroelastic model, e.g. to take liquefaction into account, a coupled BE-FE procedure seems to be the best choice. But, for such a coupled formulation also a BE formulation for simple poro must be available.

In the next section, Biot's theory is recalled and the simplification is presented. For these governing equations fundamental solutions are derived using the method of Hörmander [21]. The next step, to establish a BE formulation is straightforward following exactly the procedure given for the general Biot equations [18, 22]. After presenting this formulation a 1-d analytical solution is derived for comparison with the proposed BE formulation.

Throughout this paper, the summation convention is applied over repeated indices and Latin indices receive the values $1, 2$, and $1, 2, 3$ in 2-d and 3-d, respectively. Commas $()_{,i}$ denote spatial derivatives and, as usual, the Kronecker delta is denoted by δ_{ij}.

2. BIOT'S THEORY—GOVERNING EQUATIONS

Following Biot's approach to model the behaviour of porous media, an elastic skeleton with a statistical distribution of interconnected pores is considered [23]. This porosity is denoted by

$$\phi = \frac{V^{\mathrm{f}}}{V} \tag{1}$$

where V^{f} is the volume of the interconnected pores contained in a sample of bulk volume V. Contrary to these pores the sealed pores will be considered as part of the solid. Full saturation is assumed leading to $V = V^{\mathrm{f}} + V^{\mathrm{s}}$ with V^{s} the volume of the solid, i.e. a two-phase material is given.

One possible representation of poroelastic constitutive equation is obtained using the total stress $\sigma_{ij} = \sigma^{\mathrm{s}}_{ij} + \sigma^{\mathrm{f}}\delta_{ij}$ and the pore pressure p as independent variables [4]. Introducing Biot's effective stress coefficient α and the solid displacement u_i the constitutive equation reads

$$\sigma_{ij} = G(u_{i,j} + u_{j,i}) + (K - \tfrac{2}{3}G)u_{k,k}\delta_{ij} - \alpha\delta_{ij}p \tag{2}$$

with the shear modulus and the compression modulus of the solid frame G and K, respectively. In this equation, a linear strain displacement relation is used, i.e. small deformation gradients are assumed. Additional to the total stress σ_{ij}, as a second constitutive equation the variation of fluid volume per unit reference volume ζ is introduced

$$\zeta = \alpha u_{k,k} + \frac{\phi^2}{R}p \tag{3}$$

with material constant R. This variation of fluid ζ is defined by the mass balance over a reference volume, i.e. by the continuity equation

$$\frac{\partial \zeta}{\partial t} + q_{i,i} = a \tag{4}$$

with the specific flux q_i and a source term $a(t)$.

Further, the balance of momentum for the bulk material must be fulfilled. This dynamic equilibrium is given by

$$\sigma_{ij,j} + F_i = \varrho \frac{\partial^2 u_i}{\partial t^2} + \phi \varrho_{\mathrm{f}} \frac{\partial w_i}{\partial t} \tag{5}$$

with the bulk body force per unit volume F_i and the bulk density $\varrho = \varrho_{\mathrm{s}}(1-\phi) + \phi\varrho_{\mathrm{f}}$ (ϱ_{s} and ϱ_{f} denotes the solid and fluid density, respectively).

Next, the fluid transport in the interstitial space expressed by the specific flux $q_i = \phi w_i$ is modelled with a generalized Darcy's law

$$\phi w_i = q_i = -\kappa \left(p_{,i} + \varrho_{\mathrm{f}} \frac{\partial^2 u_i}{\partial t^2} + \frac{\varrho_{\mathrm{a}} + \phi\varrho_{\mathrm{f}}}{\phi} \frac{\partial w_i}{\partial t} \right) \tag{6}$$

where κ denotes the permeability and w_i the seepage velocity. In Equation (6), an additional density the apparent mass density ϱ_{a} is introduced by Biot [5] to describe the interaction between fluid and skeleton.

The five equations (2)–(6) represent Biot's linear theory of a poroelastic continuum. To eliminate in these five equations the seepage velocity w_i, Darcy's law has to be rearranged to find an expression for the seepage velocity. Obviously, due to the different time derivatives of w_i this is not possible in time domain. However, if the inertia effects of the fluid can be neglected, i.e. $\partial w_i / \partial t$ can be set to zero in (5) and (6), the elimination of the seepage velocity is possible. This results in the simplified dynamic equilibrium

$$\sigma_{ij,j} + F_i = \varrho \frac{\partial^2 u_i}{\partial t^2} \tag{7}$$

and the simplified dynamic version of Darcy's law

$$\phi w_i = q_i = -\kappa \left(p_{,i} + \varrho_{\mathrm{f}} \frac{\partial^2 u_i}{\partial t^2} \right) \tag{8}$$

Now, Darcy's law (8) can be used to replace the seepage velocity in the above equations (2)–(4). Rearranging them yields the governing set of differential equations for the unknowns solid displacement u_i and pore pressure p

$$G u_{i,jj} + \left(K + \frac{1}{3} G \right) u_{j,ij} - \alpha p_{,i} - \varrho \frac{\partial^2 u_i}{\partial t^2} = -F_i \tag{9a}$$

$$\kappa p_{,ii} - \frac{\phi^2}{R} \frac{\partial p}{\partial t} - \alpha \frac{\partial u_{i,i}}{\partial t} + \kappa \varrho_{\mathrm{f}} \frac{\partial^2 u_{i,i}}{\partial t^2} = -a \tag{9b}$$

This simplification and, subsequent, the possibility to represent the governing equations with this reduced set of unknowns has been published by Zienkiewicz [17]. There, the authors discussed with the help of an analytical 1-d example the limitations of this simplification. Summarizing their results, in soil mechanics or geomechanical applications with mostly low frequency acceleration the complete Biot theory does not significantly differ from the simplified form.

In the next section, fundamental solutions for the simplified Biot's equations are derived. These solutions will be later used in a convolution quadrature-based BE formulation. Therefore, it is sufficient and to the authors knowledge the only possible way to deduce the fundamental solutions in Laplace domain. To do so, first, the set of governing equations (9) is transformed to Laplace domain, denoted by $\mathscr{L}\{f(t)\}=\hat{f}(s)$ with the complex Laplace variable s. Further, vanishing initial conditions are assumed. This leads in operator notation to

$$\mathbf{B}\begin{bmatrix}\hat{u}_j\\ \hat{p}\end{bmatrix}=-\begin{bmatrix}\hat{F}_i\\ \hat{a}\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}(G\nabla^2-s^2\varrho)\delta_{ij}+(K+\frac{1}{3}G)\partial_i\partial_j & -\alpha\partial_i\\ -s(\alpha-s\kappa\varrho_{\mathrm{f}})\partial_j & \kappa\nabla^2-\dfrac{\phi^2 s}{R}\end{bmatrix} \tag{10}$$

with the not self-adjoint operator $\mathbf{B}$. Based on these equations in the next section fundamental solutions are derived.

3. FUNDAMENTAL SOLUTIONS

A fundamental solution is mathematically spoken a solution of the equation $\mathbf{BG}+\mathbf{I}\delta(\mathbf{x}-\mathbf{y})=\mathbf{0}$ where the matrix of fundamental solutions is denoted by $\mathbf{G}$, the identity matrix by $\mathbf{I}$, and the Dirac distribution by $\delta(\mathbf{x}-\mathbf{y})$. Physically interpreted the solution at point $\mathbf{x}$ due to a single force and source at point $\mathbf{y}$ is looked for.

For Biot's theory in its complete form fundamental solutions in Laplace domain are available [24]. Also, in time domain such solutions has been developed, however, not in closed form [25, 26]. As the simple poro formulation results from a simplification of Biot's theory there is a hope to find fundamental solutions by introducing these simplifications in the known fundamental solutions of Biot's complete theory. Unfortunately, the mathematical operator in (10) is too different, so that new fundamental solutions have to be calculated. But, the operator type is still an elliptical operator so the same method as for Biot's theory to find the fundamental solutions, the method of Hörmander [21], can be used.

The idea of this method is to reduce the operator given in (10) to well known operators. An overview of this method is found in the original work by Hörmander [21] and more exemplary in References [18, 27]. Following this idea the definition of the inverse matrix operator $\mathbf{B}^{-1}=\mathbf{B}^{\mathrm{co}}/\det(\mathbf{B})$ with the matrix of cofactors $\mathbf{B}^{\mathrm{co}}$ is used. The ansatz $\mathbf{G}=\mathbf{B}^{\mathrm{co}}\varphi$ for the matrix of fundamental solutions with an unknown scalar function φ inserted in the operator equation $\mathbf{BG}+\mathbf{I}\delta(\mathbf{x}-\mathbf{y})=\mathbf{0}$ yields to a more convenient representation of Equations (10)

$$\begin{aligned}\mathbf{BB}^{\mathrm{co}}\varphi+\mathbf{I}\delta(\mathbf{x}-\mathbf{y})=\det(\mathbf{B})\mathbf{I}\varphi+\mathbf{I}\delta(\mathbf{x}-\mathbf{y})&=\mathbf{0}\\ \rightsquigarrow\det(\mathbf{B})\varphi+\delta(\mathbf{x}-\mathbf{y})&=0\end{aligned} \tag{11}$$

With this reformulation, the search for a fundamental solution is reduced to solve the simpler scalar equation (11).

From the mathematical theory of Green's formula it is known that the fundamental solutions should satisfy the adjoint operator [28]. Opposite to elasticity the governing operator in poroelasticity is not self-adjoint. Therefore, here the solution for the adjoint operator $\mathbf{B}^{\star}$ is required.

Following formula (11), first, the determinant of the operator $\mathbf{B}^\star$ is calculated. This yields to the results

$$\text{2-d}: \quad \det \mathbf{B}^\star = \kappa G(K + \tfrac{4}{3}G)(\nabla^2 - \lambda_3^2)(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2) \tag{12}$$

$$\text{3-d}: \quad \det \mathbf{B}^\star = \kappa G^2(K + \tfrac{4}{3}G)\,(\nabla^2 - \lambda_3^2)^2(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2) \tag{13}$$

with the roots $\lambda_i,\ i = 1, 2, 3$

$$\begin{aligned}\lambda_{1,2}^2 = \frac{1}{2}\Bigg[&\frac{\phi^2 s}{\kappa R} + \frac{\alpha s(\alpha - s\varrho_{\mathrm{f}}\kappa)}{(K + \frac{4}{3}G)\kappa} + \frac{s^2\varrho}{K + \frac{4}{3}G}\\ &\pm\sqrt{\left(\frac{\phi^2 s}{\kappa R} + \frac{\alpha s(\alpha - s\varrho_{\mathrm{f}}\kappa)}{(K + \frac{4}{3}G)\kappa} + \frac{s^2\varrho}{(K + \frac{4}{3}G)}\right)^2 - 4\frac{s^2\varrho\phi^2 s}{R(K + \frac{4}{3}G)\kappa}}\ \Bigg]\\ \lambda_3^2 = \frac{\varrho s^2}{G}&\end{aligned} \tag{14}$$

The scalar equation corresponding to (11) becomes

$$(\nabla^2 - \lambda_3^2)(\nabla^2 - \lambda_1^2)(\nabla^2 - \lambda_2^2)\psi + \delta(\mathbf{x} - \mathbf{y}) = 0 \tag{15}$$

using an appropriate abbreviation ψ for every operator, i.e.

$$\begin{aligned}&\text{2-d}: \quad \psi = G\kappa(K + \tfrac{4}{3}G)\varphi\\ &\text{3-d}: \quad \psi = G^2\kappa(K + \tfrac{4}{3}G)(\nabla^2 - \lambda_3^2)\varphi\end{aligned} \tag{16}$$

The solution of the modified higher order Helmholtz equation (15) is

$$\text{2-d}: \quad \psi = \frac{1}{2\pi}\left[\frac{K_0(\lambda_1 r)}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} + \frac{K_0(\lambda_2 r)}{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)} + \frac{K_0(\lambda_3 r)}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)}\right] \tag{17}$$

$$\text{3-d}: \quad \psi = \frac{1}{4\pi r}\left[\frac{\mathrm{e}^{-\lambda_1 r}}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} + \frac{\mathrm{e}^{-\lambda_2 r}}{(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2)} + \frac{\mathrm{e}^{-\lambda_3 r}}{(\lambda_3^2 - \lambda_2^2)(\lambda_3^2 - \lambda_1^2)}\right] \tag{18}$$

with the zero order modified Bessel function of second kind $K_0(z)$. Further, the distance between the two points $\mathbf{x}$ and $\mathbf{y}$ is denoted by $r = |\mathbf{x} - \mathbf{y}|$.

Having in mind that the Laplace transformation of the function which describes a travelling wave front with constant speed c is $\mathrm{e}^{-rs/c} = \mathscr{L}\{H(t - r/c)\}$ (in 3-d), it is obvious that the above solution (18) represents three waves. The roots λ_1, λ_2, and λ_3 correspond to the wave velocities of the slow and fast compressional wave and to the shear wave, respectively. Comparing the fundamental solutions of Biot's complete theory the same solution is found but different λ_i are calculated. This is essentially the only but important difference. As the roots λ_i are functions of s, here, the compressional wave speeds are time dependent representing the attenuation in a poroelastic continuum. Contrary to the general Biot theory, the shear wave speed is no longer time dependent, i.e. not attenuated. The term s of λ_3 in (14) belongs to the exponential function e^{-rs/c_2} and not to the wave velocity c_2. The same is true in 2-d where the damped wave fronts are represented in Laplace domain by the modified Bessel functions $K_0(z)$.

The next steps are to insert the solution ψ back in the definition $\mathbf{G}=\mathbf{B}^{\mathrm{co}}\varphi$ taking into account the proper relation (16) between φ and ψ. After calculating the respective matrix of cofactors $\mathbf{B}^{\mathrm{co}}$ the fundamental solutions are found

$$\mathbf{G}=\begin{bmatrix}\hat{U}_{ij}^{\mathrm{s}} & \hat{U}_{i}^{\mathrm{f}}\\ \hat{P}_{j}^{\mathrm{s}} & \hat{P}^{\mathrm{f}}\end{bmatrix}=\frac{1}{G\kappa(K+\frac{4}{3}G)}\begin{bmatrix}(F\nabla^2+AD)\delta_{ij}-F\partial_{ij} & -A\alpha\partial_i\\ -AE\partial_i & A(B\nabla^2+A)\end{bmatrix}\psi \tag{19}$$

with the abbreviations $A=G\nabla^2-s^2\varrho$, $B=(K+\frac{1}{3}G)$, $D=\kappa\nabla^2-\phi^2 s/R$, $E=s(\alpha-\varrho_{\mathrm{f}}\kappa)$, $F=BD-\alpha E$. The difference of the 2-d solution and the 3-d solution lies only in different functions ψ from (17) or (18), respectively. The explicit expressions for the fundamental solutions can be found in Appendix A.

To visualize the differences in the simple poro and Biot's complete theory, the fundamental solution U_{00}^{s} in 2-d and in 3-d is considered in Figure 1 using the material data of a soil (see Table I). The results are plotted for $r=1$ m. In Figure 1, not the frequency dependent solution is plotted but the time response caused by a unit step loading $H(t)$. This result is obtained using the convolution quadrature method (see References [18–20]) to solve the convolution integral between the time domain fundamental solution U_{00}^{s} and the load.

The first deviation from zero ($t\approx 0.0002$ s) represents the fast compressional wave and the larger effect ($t\approx 0.0012$ s) is caused by the shear wave. In both, 2-d and 3-d, no significant differences are visible except at the jump of the shear wave which arrival time is slightly later for the simple poro than for the complete theory. Also, the oscillations around this jump are more pronounced. As the time step size used for computing these results is the same for both theories, it can be concluded that the fundamental solutions of simple poro are numerically more involved. This may be caused by the fact that in this case the shear wave is no longer damped.

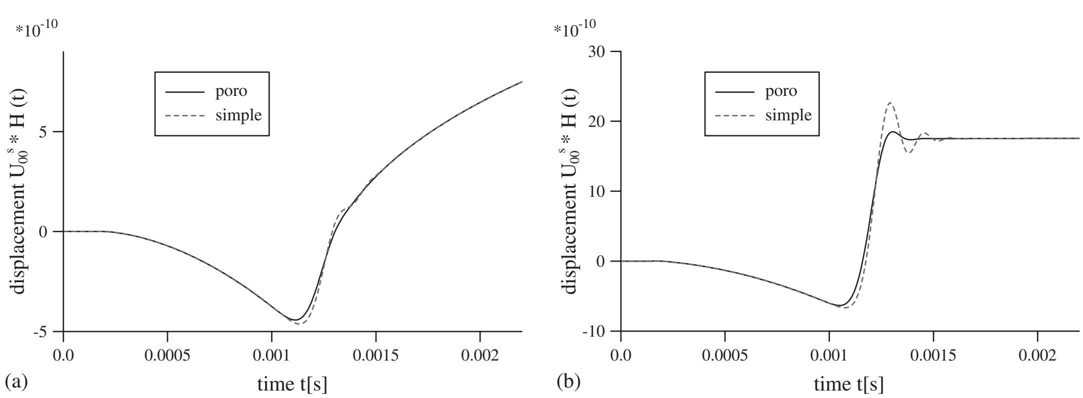

Figure 1. Fundamental solution U_{00}^{s} convoluted with $H(t)$ versus time: (a) 2-d; and (b) 3-d.

Table I. Material data of Berea sandstone (rock) and water saturated soil.

	E (N/m^2)	ν	ρ (kg/m^3)	ρ_{f} (kg/m^3)	ϕ	R (N/m^2)	α	κ (m^4/N s)
Rock	1.44×10^{10}	0.0	2458	1000	0.19	4.7×10^8	0.86	1.9×10^{-10}
Soil	2.544×10^8	0.298	1884	1000	0.48	1.2×10^9	0.98	3.55×10^{-9}

4. BOUNDARY ELEMENT FORMULATION

The derivation of the BE formulation follows the usual procedure starting from a weighted residual statement. After two partial integrations with respect to the spatial variable the boundary integral equation is achieved. As this procedure is extensively described in References [18, 22], here, only the differences are given.

The poroelastodynamic integral equation can be derived directly by equating the inner product of the fundamental solutions $\mathbf{G}$ (19) and the set of governing equations (10) to a null vector, i.e.

$$\int_\Omega \mathbf{G}^{\mathrm{T}} \mathbf{B} \begin{bmatrix} \hat{u}_j \\ \hat{p} \end{bmatrix} \mathrm{d}\Omega = \mathbf{0} \tag{20}$$

where the integration is performed over a domain Ω with boundary Γ and vanishing body forces F_i and sources a are assumed. By this inner product, essentially, the error in satisfying the governing differential equations (10) is forced to be orthogonal to $\mathbf{G}$. According to the theory of Green's formula and using partial integration the operator $\mathbf{B}$ is transformed from acting on the vector of unknowns $[\hat{u}_j \;\; \hat{p}]^{\mathrm{T}}$ to the matrix of fundamental solutions $\mathbf{G}$. These steps yields the following system of integral equations given in matrix notation as

$$\int_\Gamma \begin{bmatrix} \hat{U}^{\mathrm{s}}_{ij} & -\hat{P}^{\mathrm{s}}_j \\ \hat{U}^{\mathrm{f}}_i & -\hat{P}^{\mathrm{f}} \end{bmatrix} \begin{bmatrix} \hat{t}_i \\ \hat{q} \end{bmatrix} \mathrm{d}\Gamma - \int_\Gamma \begin{bmatrix} \hat{T}^{\mathrm{s}}_{ij} & \hat{Q}^{\mathrm{s}}_j \\ \hat{T}^{\mathrm{f}}_i & \hat{Q}^{\mathrm{f}} \end{bmatrix} \begin{bmatrix} \hat{u}_i \\ \hat{p} \end{bmatrix} \mathrm{d}\Gamma = \begin{bmatrix} \hat{u}_j \\ \hat{p} \end{bmatrix} \tag{21}$$

In both integrations by parts, the divergence theorem and the filter property of the Dirac distribution is used. Additionally, the traction vector $\hat{t}_i = \hat{\sigma}_{ij} n_j$ and the normal flux $\hat{q} = -\kappa(\hat{p}_{,i} + \varrho_{\mathrm{f}} s^2 \hat{u}_i) n_i$ is introduced, and the abbreviations

$$\hat{T}^{\mathrm{s}}_{ij} = \left[((K - \tfrac{2}{3}G)\hat{U}^{\mathrm{s}}_{kj,k} + \alpha s \hat{P}^{\mathrm{s}}_j)\delta_{i\ell} + G(\hat{U}^{\mathrm{s}}_{ij,\ell} + \hat{U}^{\mathrm{s}}_{\ell j,i}) \right] n_\ell \tag{22a}$$

$$\hat{Q}^{\mathrm{s}}_j = \kappa \hat{P}^{\mathrm{s}}_{j,i} n_i \tag{22b}$$

$$\hat{T}^{\mathrm{f}}_i = \left[((K - \tfrac{2}{3}G)\hat{U}^{\mathrm{f}}_{k,k} + \alpha s \hat{P}^{\mathrm{f}})\delta_{i\ell} + G(\hat{U}^{\mathrm{f}}_{i,\ell} + \hat{U}^{\mathrm{f}}_{\ell,i}) \right] n_\ell \tag{22c}$$

$$\hat{Q}^{\mathrm{f}} = \kappa \hat{P}^{\mathrm{f}}_{,i} n_i \tag{22d}$$

are used, where (22a) and (22b) can be interpreted as being the adjoint term to the traction vector $\hat{t}_i$ and the flux $\hat{q}$, respectively. In the definition of the flux $\hat{q}$ the simplified version of Darcy's law (8) is used. However, in its corresponding fundamental solution $\hat{Q}^{\mathrm{f}}$ and in the adjoint term $\hat{Q}^{s}_{j}$ only a quasi-static version of Darcy's law is found. This is due to the neglect of the inertia effects in the fluid.

When moving $\mathbf{y}$ to the boundary Γ to determine the unknown boundary data, it is necessary to know the behaviour of the fundamental solutions when $r=|\mathbf{y}-\mathbf{x}|$ tends to zero, i.e. when an integration point $\mathbf{x}$ approaches a collocation point $\mathbf{y}$. Six of the eight fundamental solutions, four in $\mathbf{G}$ and four calculated by Equations (22), are singular. The order of their singularity can be determined by series representations with respect to the variable r. This variable is found in these solutions either in the exponential function in the 3-d solutions or in the Bessel functions in case of 2-d. Else, only powers of r appear. So, it is sufficient to insert in the fundamental solutions (A1a), (A3a)–(A1d), (A3d) and (A2a), (A4a)–(A2d), (A4d) the following series expansions:

$$e^{-\lambda_k s r} = \sum_{\ell=0}^{\infty} \frac{(-\lambda_k s r)^{\ell}}{\ell!} = 1 - \lambda_k s r + \lambda_k^2 s^2 r^2 + \mathcal{O}(r^3) \tag{23}$$

for the exponential function, and for the Bessel functions:

$$K_0(\lambda_k s r) = -(\ln(\lambda_k s r) - \ln 2 + \gamma) + \mathcal{O}(r^2) \tag{24a}$$

$$K_1(\lambda_k s r) = \frac{1}{\lambda_k s r} + \frac{\lambda_k s r}{2}\left(\ln(\lambda_k s r) - \ln 2 + \gamma - \frac{1}{2}\right) + \mathcal{O}(r^3) \tag{24b}$$

$$\gamma = \lim_{n\to\infty}\left(\sum_{v=1}^{n}\frac{1}{v} - \ln n\right) \approx 0.577216 \quad \text{(Euler-constant)}$$

Inserting these series in the fundamental solutions and a subsequent ordering with respect to the power of r yields to the singular behaviour. This leads for 3-d to

$$\hat{P}^{s}_{i}, \hat{U}^{f}_{i} = \mathcal{O}(r^0) \tag{25a}$$

$$\hat{U}^{s}_{ij} = \underbrace{\frac{1+v}{8\pi E(1-v)}\{r_{,i}r_{,j} + \delta_{ij}(3-4v)\}\frac{1}{r}}_{\text{elastostatic fundamental solution}} + \mathcal{O}(r^0) \tag{25b}$$

$$\hat{P}^{f} = \frac{1}{4\pi\kappa}\frac{1}{r} + \mathcal{O}(r^0) \tag{25c}$$

$$\hat{Q}^{s}_{j} = \frac{1+v}{8\pi E(1-v)}\,\alpha(1-2v)(r_{,n}r_{,j} - n_j)\frac{1}{r} + \mathcal{O}(r^0) \tag{25d}$$

$$\hat{T}^{f}_{i} = \frac{1}{8\pi\kappa}\left\{s(\alpha - s\varrho_f\kappa)\frac{1-2v}{1-v}r_{,i}r_{,n} + n_i s\frac{\alpha + s\varrho_f\kappa(1-2v)}{1-v}\right\}\frac{1}{r} + \mathcal{O}(r^0) \tag{25e}$$

$$\hat{T}^{s}_{ij} = \underbrace{\frac{-1}{8\pi(1-v)}\{[(1-2v)\delta_{ij} + 3r_{,i}r_{,j}]r_{,n} - (1-2v)(r_{,j}n_i - r_{,i}n_j)\}\frac{1}{r^2}}_{\text{elastostatic fundamental solution}} + \mathcal{O}(r^0) \tag{25f}$$

$$\hat{Q}^{\mathrm{f}} = \underbrace{-\frac{r_{,n}}{4\pi r^2}}_{\text{acoustic fundamental solution}} + \mathcal{O}(r^0) \tag{25g}$$

and for 2-d to

$$\hat{P}_i^{\mathrm{s}}, \hat{U}_i^{\mathrm{f}} = \mathcal{O}(r^0) \tag{26a}$$

$$\hat{U}_{ij}^{\mathrm{s}} = \underbrace{-\frac{1+\nu}{4\pi E(1-\nu)}\{(3-4\nu)\ln(r)\delta_{ij} - r_{,i}r_{,j}\}}_{\text{elastostatic fundamental solution}} + \mathcal{O}(r^0) \tag{26b}$$

$$\hat{P}^{\mathrm{f}} = -\frac{1}{2\pi\kappa}\ln(r) + \mathcal{O}(r^0) \tag{26c}$$

$$\hat{Q}_j^{\mathrm{s}} = \frac{n_j(1+\nu)}{4\pi}\frac{\alpha(1-2\nu)}{E(1-\nu)}\ln(r) + \mathcal{O}(r^0) \tag{26d}$$

$$\hat{T}_i^{\mathrm{f}} = -\frac{n_i s}{4\pi\kappa(1-\nu)}\{s\varrho_{\mathrm{f}}\kappa + \alpha(1-2\nu)\}\ln(r) + \mathcal{O}(r^0) \tag{26e}$$

$$\hat{T}_{ij}^{\mathrm{s}} = \underbrace{\frac{-2r_{,n}r_{,i}r_{,j} + (1-2\nu)(r_{,n}\delta_{ij} + n_j r_{,i} - n_i r_{,j})}{4\pi(1-\nu)r}}_{\text{elastostatic fundamental solution}} + \mathcal{O}(r^0) \tag{26f}$$

$$\hat{Q}^{\mathrm{f}} = \underbrace{-\frac{r_{,n}}{2\pi r}}_{\text{acoustic fundamental solution}} + \mathcal{O}(r^0) \tag{26g}$$

In Equations (25) and (26), it is shown that the fundamental solutions are either regular (25a) and (26a), weakly singular (25b), (26b)–(25e), (26e) or strongly singular (25f), (26f) and (25g), (26g). The strongly singular parts in the kernel functions (25f), (26f) and (25g), (26g) are known from elastostatics and acoustics, respectively.

Therefore, shifting in (22) point $\mathbf{y}$ to the boundary Γ results in the boundary integral equation

$$\int_\Gamma \begin{bmatrix} \hat{U}_{ij}^{\mathrm{s}} & -\hat{P}_j^{\mathrm{s}} \\ \hat{U}_i^{\mathrm{f}} & -\hat{P}^{\mathrm{f}} \end{bmatrix} \begin{bmatrix} \hat{t}_i \\ \hat{q} \end{bmatrix} \mathrm{d}\Gamma = \oint_\Gamma \begin{bmatrix} \hat{T}_{ij}^{\mathrm{s}} & \hat{Q}_j^{\mathrm{s}} \\ \hat{T}_i^{\mathrm{f}} & \hat{Q}^{\mathrm{f}} \end{bmatrix} \begin{bmatrix} \hat{u}_i \\ \hat{p} \end{bmatrix} \mathrm{d}\Gamma + \begin{bmatrix} c_{ij} & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} \hat{u}_i \\ \hat{p} \end{bmatrix} \tag{27}$$

with the integral free terms c_{ij} and c known from elastostatics and acoustics, respectively, and with the Cauchy principal value integral $\oint$. A transformation to time domain gives, finally, the time dependent integral equation for simplified poroelasticity

$$\int_0^t \int_\Gamma \begin{bmatrix} U_{ij}^{\mathrm{s}}(t-\tau, \mathbf{y}, \mathbf{x}) & -P_j^{\mathrm{s}}(t-\tau, \mathbf{y}, \mathbf{x}) \\ U_i^{\mathrm{f}}(t-\tau, \mathbf{y}, \mathbf{x}) & -P^{\mathrm{f}}(t-\tau, \mathbf{y}, \mathbf{x}) \end{bmatrix} \begin{bmatrix} t_i(\tau, \mathbf{x}) \\ q(\tau, \mathbf{x}) \end{bmatrix} \mathrm{d}\Gamma\, \mathrm{d}\tau$$

$$
= \int_0^t \oint_\Gamma \begin{bmatrix} T_{ij}^{\mathrm{s}}(t-\tau,\mathbf{y},\mathbf{x}) & Q_j^{\mathrm{s}}(t-\tau,\mathbf{y},\mathbf{x}) \\ T_i^{\mathrm{f}}(t-\tau,\mathbf{y},\mathbf{x}) & Q^{\mathrm{f}}(t-\tau,\mathbf{y},\mathbf{x}) \end{bmatrix} \begin{bmatrix} u_i(\tau,\mathbf{x}) \\ p(\tau,\mathbf{x}) \end{bmatrix} \mathrm{d}\Gamma\,\mathrm{d}\tau
+ \begin{bmatrix} c_{ij}(\mathbf{y}) & 0 \\ 0 & c(\mathbf{y}) \end{bmatrix} \begin{bmatrix} u_i(t,\mathbf{y}) \\ p(t,\mathbf{y}) \end{bmatrix} \tag{28}
$$

A boundary element formulation is achieved following the usual procedure. First, the boundary surface Γ is discretized by E iso-parametric elements Γ_e where F polynomial shape functions $N_e^f(\mathbf{x})$ are defined. Hence, the following ansatz functions are used with the time-dependent nodal values $u_i^{ef}(t), t_i^{ef}(t), p^{ef}(t)$, and $q^{ef}(t)$

$$
\begin{aligned}
u_i(\mathbf{x},t) &= \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f(\mathbf{x})u_i^{ef}(t), \quad t_i(\mathbf{x},t) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f(\mathbf{x})t_i^{ef}(t) \\
p(\mathbf{x},t) &= \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f(\mathbf{x})p^{ef}(t), \quad q(\mathbf{x},t) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f(\mathbf{x})q^{ef}(t)
\end{aligned} \tag{29}
$$

In Equations (29), the shape functions of all four variables are denoted by the same function $N_e^f(\mathbf{x})$ indicating the same approximation level of all variables. This is not mandatory but usual (for mixed shape functions see Reference [29]).

Next, a time discretization has to be introduced. Since no time-dependent fundamental solutions are known, the convolution quadrature method (see References [18–20]) is used. Hence, after dividing the time period t in N intervals of equal duration Δt, i.e. $t = N\Delta t$, the convolution integrals between the fundamental solutions and the nodal values are approximated by the convolution quadrature method. This together with the spatial discretization results in the following boundary element time stepping formulation for $n = 0, 1, \ldots, N$

$$
\begin{aligned}
\begin{bmatrix} c_{ij} & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} u_i(\mathbf{y},n\Delta t) \\ p(\mathbf{y},n\Delta t) \end{bmatrix} = \sum_{e=1}^{E}\sum_{f=1}^{F}\sum_{k=0}^{n} \Biggl\{ & \begin{bmatrix} \omega_{n-k}^{ef}(\hat{U}_{ij}^{\mathrm{s}},\mathbf{y},\Delta t) & -\omega_{n-k}^{ef}(\hat{P}_j^{\mathrm{s}},\mathbf{y},\Delta t) \\ \omega_{n-k}^{ef}(\hat{U}_i^{\mathrm{f}},\mathbf{y},\Delta t) & -\omega_{n-k}^{ef}(\hat{P}^{\mathrm{f}},\mathbf{y},\Delta t) \end{bmatrix} \begin{bmatrix} t_i^{ef}(k\Delta t) \\ q^{ef}(k\Delta t) \end{bmatrix} \\
& - \begin{bmatrix} \omega_{n-k}^{ef}(\hat{T}_{ij}^{\mathrm{s}},\mathbf{y},\Delta t) & \omega_{n-k}^{ef}(\hat{Q}_j^{\mathrm{s}},\mathbf{y},\Delta t) \\ \omega_{n-k}^{ef}(\hat{T}_i^{\mathrm{f}},\mathbf{y},\Delta t) & \omega_{n-k}^{ef}(\hat{Q}^{\mathrm{f}},\mathbf{y},\Delta t) \end{bmatrix} \begin{bmatrix} u_i^{ef}(k\Delta t) \\ p^{ef}(k\Delta t) \end{bmatrix} \Biggr\}
\end{aligned} \tag{30}
$$

with the weights, e.g.

$$
\omega_{n-k}^{ef}(\hat{U}_{ij}^{\mathrm{s}},\mathbf{y},\Delta t) = \frac{\mathscr{R}^{-(n-k)}}{L} \sum_{\ell=0}^{L-1} \int_\Gamma \hat{U}_{ij}^{\mathrm{s}}\left(\frac{\gamma(\mathrm{e}^{\mathrm{i}\ell(2\pi/L)}\mathscr{R})}{\Delta t},\mathbf{y},\mathbf{x}\right) N_{\mathrm{e}}^f(\mathbf{x})\,\mathrm{d}\Gamma \mathrm{e}^{-\mathrm{i}(n-k)\ell(2\pi/L)} \tag{31}
$$

Note, the calculation of the integration weights is only based on the Laplace transformed fundamental solutions which are available. Therefore, with the time stepping procedure (30) a boundary element formulation for simplified poroelastodynamics is given without time dependent fundamental solutions.

Due to the same singular behaviour of the fundamental solutions compared to Biot's complete theory or elastostatics the well known integration procedures can be used. Further, point collocation is used to establish a system of equations. Finally, the usual recursion formula known, e.g. from elastodynamics is achieved (see, e.g. Reference [30]). For details see References [18, 22].

5. NUMERICAL EXAMPLES

In order to validate the proposed boundary element approach, two problems are investigated: first, the results achieved by the simplified BEM are compared to an analytical solution of a 1-d column, and, second, the simplified method is compared with Biot's complete theory at the example of a half space under a vertical load in 2-d and 3-d to study the effects of the simplification on wave propagation phenomenon. In the following tests, the used material data are those of a rock [31] (Berea sandstone) in case of the column and for the half space example those of a coarse water saturated soil [32]. The data for both materials are collected in Table I. In contrast to the constitutive equation (2) in Table I the Young's modulus E and Poisson's ratio v is used and not the shear modulus G and the compression modulus K because Poisson's ratio of Berea sandstone has been changed to $v=0$ to represent better the 1-d behaviour of the column in the following example.

5.1. Poroelastic column

A 1-d column of length 3 m as sketched in Figure 2 is considered. It is assumed that the side walls and the bottom are rigid, frictionless, and impermeable. Hence, the displacements normal to the surface are blocked and the column is otherwise free to slide only parallel to the wall. At the top, the total stress vector $t_y = -1\ \mathrm{N/m^2} H(t)$ and the pore pressure $p=0\ \mathrm{N/m^2}$ is a given, i.e. a normal pressure force starts acting with $t>0$ and the fluid particles are assumed to be on a free fluid surface. Due to these restrictions, the 3-d continuum is reduced to a 1-d column with the only degree of freedom in y direction. This makes it possible to deduce a semi-analytical solution for comparison with the proposed BE formulation. The derivation follows exactly the same procedure as the corresponding solution for Biot's complete theory [33]. Therefore, it is sketched only briefly.

5.1.1. Analytical solution. For the above given problem, the governing set of differential equations (10) is reduced to two scalar coupled ordinary differential equations

$$(K + \tfrac{4}{3}G)\hat{u}_{y,yy} - \alpha\hat{p}_{,y} - s^2\varrho\hat{u}_y = 0 \tag{32}$$

$$\kappa\hat{p}_{,yy} - \frac{\phi^2 s}{R}\hat{p} - (\alpha - s\varrho_{\mathrm{f}}\kappa)s\hat{u}_{y,y} = 0 \tag{33}$$

with vanishing body forces F_i and sources a. The boundary conditions in Laplace domain are

$$\hat{u}_y(y=0)=0, \quad \hat{q}_y(y=0)=0 \quad \hat{\sigma}_y(y=\ell) = -1, \quad \hat{p}(y=\ell)=0 \tag{34}$$

where an impulse function for the temporal behaviour $f(t)=\delta(t)$ is assumed, together with vanishing initial conditions. Due to the neglected body forces this is a system of homogeneous

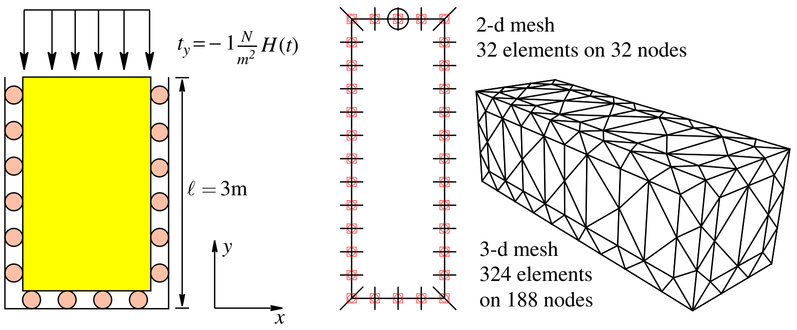

Figure 2. Geometry, boundary conditions, and discretizations of the column.

ordinary differential equations with inhomogeneous boundary conditions. Such a system can be solved by the following exponential ansatz:

$$\hat{u}_y(y) = U e^{\lambda y}, \quad \hat{p}(y) = P e^{\lambda y} \tag{35}$$

Inserting the ansatz functions (35) in Equations (32) and (33) results in an Eigenvalue problem for λ

$$\begin{bmatrix} (K + \frac{4}{3}G)\lambda^2 - \varrho s^2 & -\alpha\lambda \\ -s(\alpha - s\varrho_f \kappa)\lambda & \lambda^2 \kappa - \dfrac{\phi^2 s}{R} \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \mathbf{0} \tag{36}$$

The Eigenvalues are the four roots of (14) $-\lambda_1, +\lambda_1, -\lambda_2$, and $+\lambda_2$. These roots lead to the solution of the homogeneous problem

$$\hat{u}_y(y) = \sum_{i=1}^{4} U_i e^{\lambda_i y}, \quad \hat{p}(y) = \sum_{i=1}^{4} P_i e^{\lambda_i y} \tag{37}$$

The eight unknown constants U_i and $P_i, i = 1, \ldots, 4$, cannot be determined by the four boundary conditions (34) alone. Also none of the complex roots can be excluded due to physical reasons. But the Eigenvector of the system (36) gives the relation

$$P_i = \underbrace{\frac{(K + \frac{4}{3}G)\lambda_i^2 - \varrho s^2}{\alpha \lambda_i}}_{d_i} \cdot U_i \tag{38}$$

Finally, if solution (37) with property (38) is inserted into the one-dimensional form of the constitutive equation (2)

$$\hat{\sigma}_y(s, y) = \left(K + \frac{4}{3}G\right) \hat{u}_{y,y} - \alpha \hat{p} = \left(K + \frac{4}{3}G\right) \sum_{i=1}^{4} \lambda_i U_i e^{\lambda_i y} - \alpha \hat{p}(s, y) \tag{39}$$

and the one-dimensional form of Darcy's law (8)

$$\hat{q}_y(s,y) = -\kappa(\hat{p}_{,y} + s^2 \varrho_f \hat{u}_y) = -\frac{\kappa(K+\frac{4}{3}G)}{\alpha}\sum_{i=1}^{4}\lambda_i^2 U_i e^{\lambda_i y} + \kappa s^2\left(\frac{\varrho - \alpha\varrho_f}{\alpha}\right)\hat{u}_y(s,y) \tag{40}$$

the remaining four constants U_i can be fit to the four boundary conditions. This leads to four equations with four unknowns. Also here, with the simplified Darcy's law (40) the difference to the complete solution is obvious.

Finally, the solutions for the displacement and the pore pressure are achieved by inserting these coefficients in the ansatz functions (37)

$$\hat{u}_y = \frac{1}{(K+\frac{4}{3}G)(d_1\lambda_2 - d_2\lambda_1)}\left[d_2\frac{\sinh\lambda_1 y}{\cosh\lambda_1\ell} - d_1\frac{\sinh\lambda_2 y}{\cosh\lambda_2\ell}\right] \tag{41}$$

$$\hat{p} = \frac{d_1 d_2}{(K+\frac{4}{3}G)(d_1\lambda_2 - d_2\lambda_1)}\left[\frac{\cosh\lambda_1 y}{\cosh\lambda_1\ell} - \frac{\cosh\lambda_2 y}{\cosh\lambda_2\ell}\right] \tag{42}$$

The corresponding stress and flux is calculated with the constitutive equation (39) and Darcy's law (40), respectively. The time dependent response has to be evaluated with the convolution quadrature method, therefore, the solution is above called semi-analytical.

5.1.2. Comparison with the proposed BE formulation. To validate the proposed BE formulation the above given 1-d analytical solution is compared to a 2-d and a 3-d BE calculation. The used meshes are depicted in Figure 2. In the following, the displacement solutions are given at the midpoint of the loaded surface, i.e. in 1-d it is $y=\ell=3$ m, and the pressure solutions are given at the midpoint of the support, i.e. in 1-d at $y=0$ m. The comparison is performed in the frequency domain as well as in time domain.

In Figure 3, the absolute value of the displacement $|\hat{u}_y(\omega, y=3\,\mathrm{m})|$ at the top of the column is plotted versus frequency ω. The analytical results for Biot's theory are named 'poro 1-d' and they are compared to the simplified theory named 'simple 1-d', 'simple 2-d', and 'simple 3-d' for analytical calculation and the 2-d and 3-d BEM results, respectively. The used material data are those of a rock (Berea sandstone, see Table I). In Figure 3, clearly the first three resonance peaks are identified which do not differ for both theories. Further, the proposed BE formulation agrees very well with the analytical solution. Not shown are results of the BE formulation based on Biot's complete theory because they cannot be distinguished from the simple poro formulation.

Next, the time-dependent behaviour is discussed. In Figure 4, the time history of the displacement $u_y(t, y=3\text{ m})$ at the top of the column caused by a step stress loading $t_y(t, y=l) = -1\text{ N/m}^2 H(t)$ is depicted. The used time step size is $\Delta t = 0.0001$ s. The same comparison as shown in frequency domain is performed. As expected from the frequency domain results the solutions for Biot's complete theory and the simplified theory coincide perfectly. Also, the 2-d and 3-d BE solution agree very well with the 1-d solution. The minor differences can be minimized by adjusting the time step size closer to an optimal value. As known from the BE formulation for Biot's theory there exists a lower critical time step size. However, because this

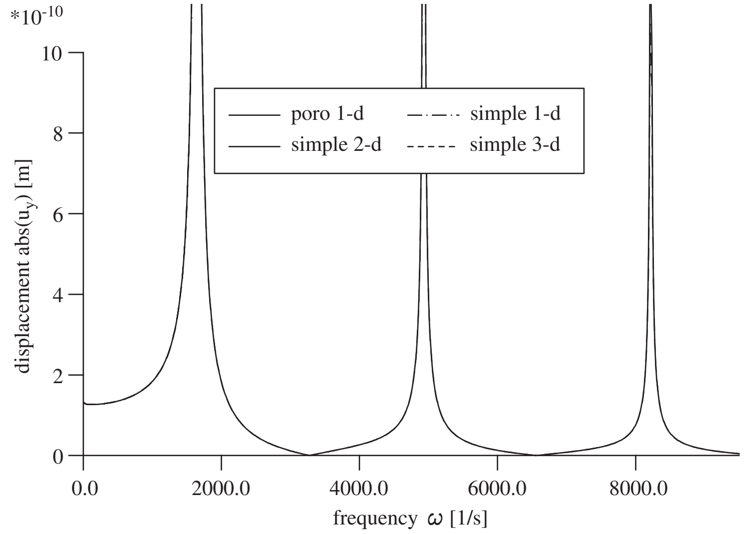

Figure 3. Displacement in y-direction at the top of the column versus frequency: comparison of analytical results with 2-d and 3-d BEM.

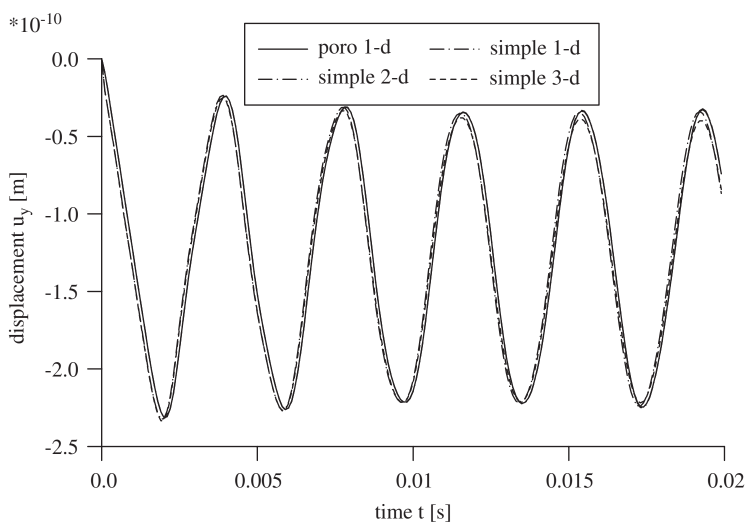

Figure 4. Displacement in y-direction at the top of the column versus time: comparison of analytical results with 2-d and 3-d BEM.

lower limit is the same for both poroelastic theories it is not studied here. For the study on this critical time step size the reader is referred to Reference [18].

Additionally to the displacement results, the pore pressure solution is presented in Figure 5. Also, in the pressure solution no significant differences between the two poroelastic models are

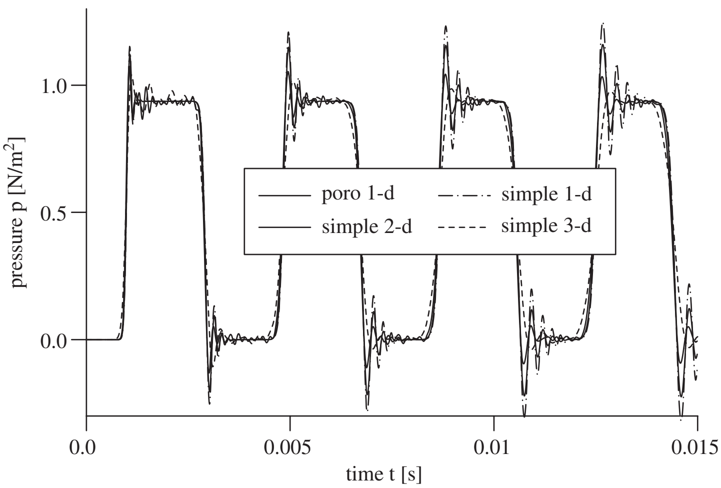

Figure 5. Pressure at the bottom of the column versus time: comparison of analytical results with 2-d and 3-d BEM.

visible. Further, the 2-d and 3-d BE solution approximate the semi-analytical result well where the overshooting at the jumps are caused obviously by the convolution quadrature. However, the non-smooth behaviour of the pressure is calculated well by the BEM where the 3-d formulation has more problems as the 2-d formulation. This is caused by the difficulties in representing the corner and edge singularities of the 3-d model which does not exist in the 1-d model. These problems are inherent in any BE formulation based on point collocation and conforming elements.

5.2. Wave propagation in a poroelastic half space

To demonstrate that the results of the u-p formulation with neglect of the derivative of the seepage velocity are similar to the results of Biot's complete theory, the displacement response and the pore pressure distribution of a poroelastic half space in 2-d and 3-d is compared, respectively. The material data in both test examples are those of a soil (see Table I).

5.2.1. 2-d model of a poroelastic half space. First, the half space is modelled in 2-d with a strip of 51 m length, where 51 linear elements are used (see Figure 6). The simulated half-space is loaded by a vertical total stress vector $t_y = -1000\,\mathrm{N/m^2}(H(t) - H(\Delta t))$ at an area of 1 m and the remaining surface is traction free. The load simulates an impulse by keeping the load over one time step. The free surface is assumed to be permeable, i.e. the pore pressure is zero all over the surface.

First, the time history of the displacement at point A is presented. In Figure 7, the calculated horizontal and vertical displacement at point A is plotted versus time for both formulations. As before, the u-p formulation with neglect of the derivative of the seepage velocity are denoted 'simplified poro' and the original Biot u-p formulation is denoted 'poro'. Clearly, the arrival of the fast compressional wave at $t \approx 0.01$ s and of the Rayleigh wave at $t \approx 0.09$ s can be observed. As expected the slow compressional wave is not visible due to the dispersion

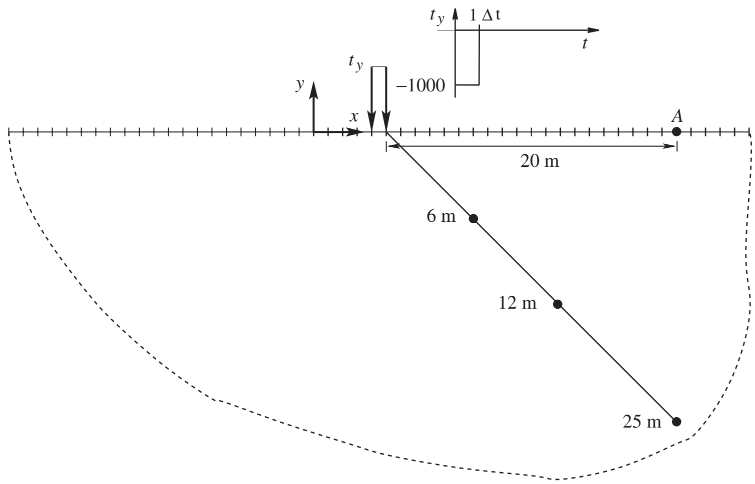

Figure 6. Poroelastic half space in 2-d: mesh and loading.

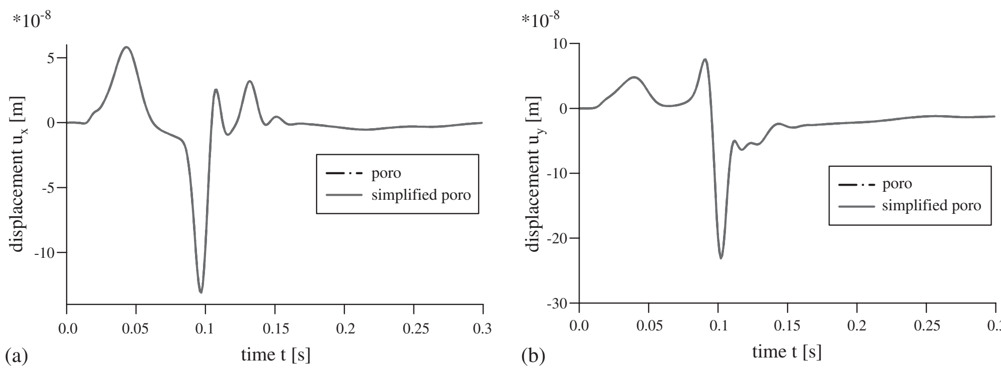

Figure 7. Vertical and horizontal displacement at point A: (a) horizontal; and (b) vertical.

effects and the shear wave is covered by the Rayleigh wave. In both co-ordinate directions no differences are visible between both formulations. The differences in the displacement amplitudes are approximately of the order $\mathcal{O}(10^{-3})$.

Additionally the pore pressure distribution under the surface is observed by variation of the depth from −6 to −20 m. The various locations are depicted in Figure 6. The time histories of the pore pressure are presented in Figure 8 for both formulations. As before in the displacement results, no significant differences between the simplified formulation and Biot's equations are found. In all three depths the arrival of the fast compressional wave is observed as a more or less wide peak. After some oscillations of the numerical solution the pore pressure decreases to zero as expected result for an impulse load.

5.2.2. 3-d model of a poroelastic half space. For the 3-d model of the half space a strip of 33 m × 6 m has been discretized with 396 triangular linear elements on 238 nodes (see Figure 9). Different to the 2-d simulation, the half space is loaded by a vertical total stress

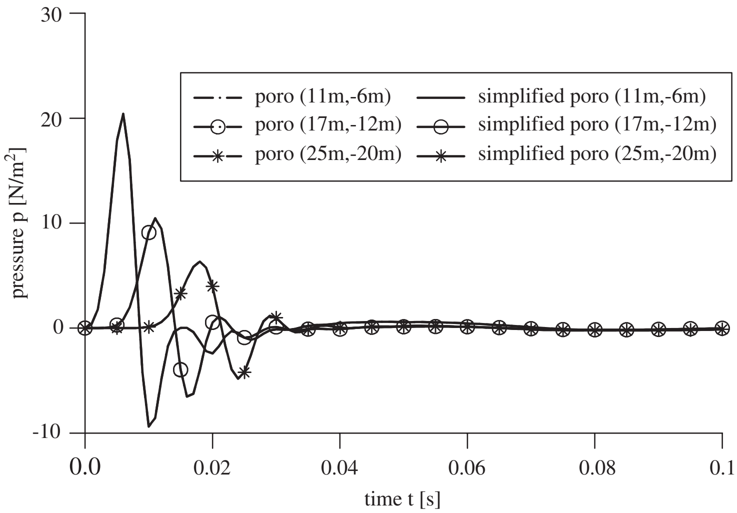

Figure 8. Pore pressure distribution below the surface at different points.

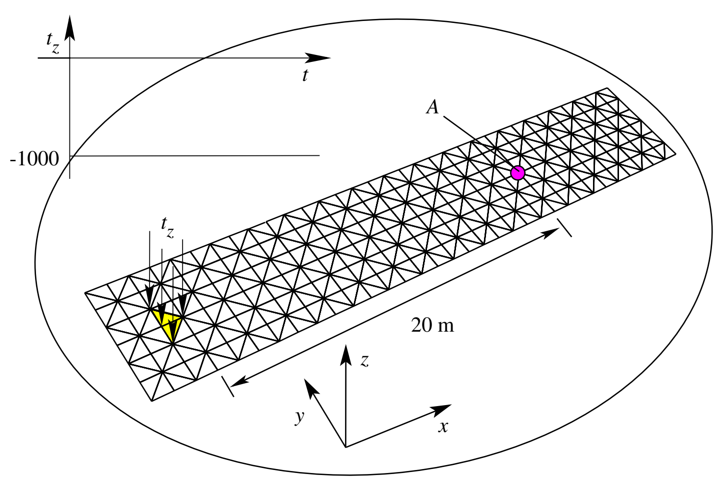

Figure 9. Poroelastic half space in 3-d: mesh and loading.

vector $t_z = -1000\,\mathrm{N/m^2}\,H(t)$ at an area of $1\,\mathrm{m}^2$ which is kept constant over the whole observation period. The remaining surface is traction free and assumed to be permeable, i.e. the pore pressure is zero all over the surface.

In Figure 10, the calculated horizontal and vertical displacement is plotted versus time at point A. Different to the 2-d example in 3-d some differences between the simplified theory and Biot's theory are visible. However, these differences are very small and in the range which can also be affected by numerics, i.e. also a change in the time step size can result in differences

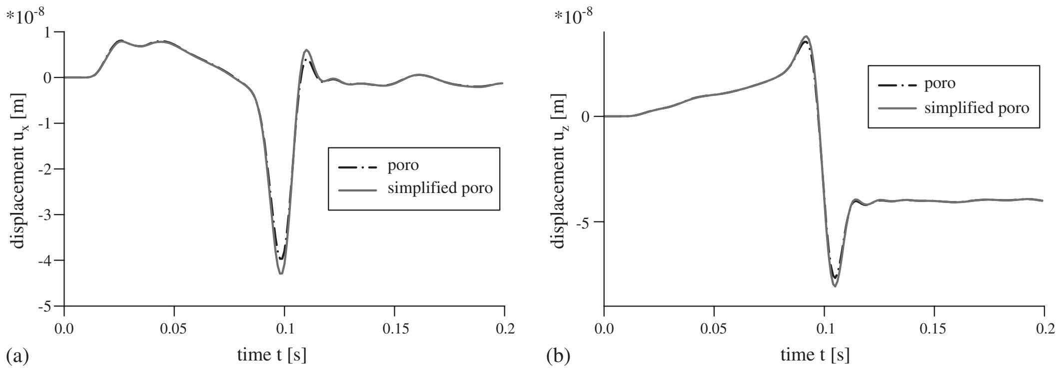

Figure 10. Vertical and horizontal displacement at point A: (a) horizontal; and (b) vertical.

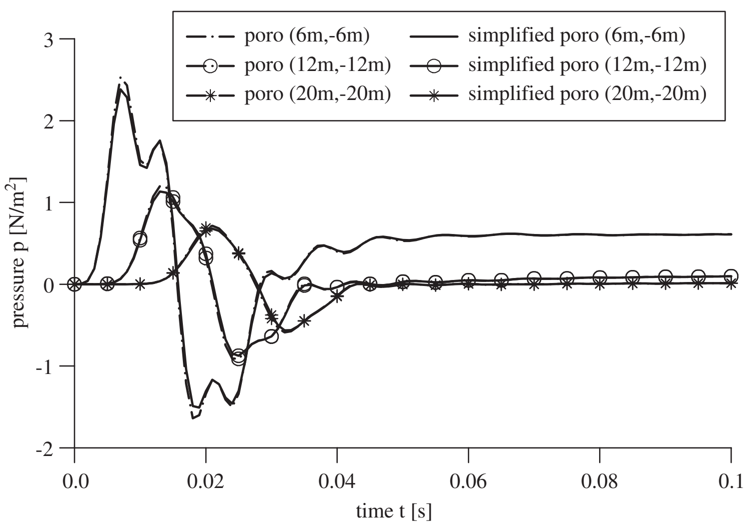

Figure 11. Pore pressure distribution below point A.

of the same order. So, in principle it can be concluded that also in the 3-d calculation both formulations give the same result.

The pore pressure distribution in different depths comparable to the study in 2-d is presented in Figure 11. There, the pore pressure is depicted versus time in a depth of 6, 12, and 20 m. Due to the larger distance from the excitation point the fast compressional wave needs different times to reach the chosen points. Also different to the 2-d calculation the pore pressure does not vanish after the passage of the wave because the load is kept over the total observation period. Further, the pore pressure reduces with increasing depth as expected.

Finally, this comparison shows that the simplified theory can be used for the chosen material, a soil and a rock, and the presented excitations. There is no significant difference to Biot's complete theory. This confirms the results presented in Reference [17].

6. CONCLUSIONS

Based on Biot's theory, a poroelastodynamic boundary element formulation with neglected derivative of the seepage velocity is presented for analysing wave propagation in two- and three-dimensional saturated porous continua. For different examples, this formulation has been compared with a BE formulation based on Biot's complete theory. A 1-d column was investigated analytically and compared with the approximated results of the simplified poroelastic solution, and a half space under a vertical load was considered for studying the difference between the complete u-p formulation and the formulation with omitting the derivative of the seepage velocity. For the investigated materials the solution from the complete u-p formulation and from the simplified poroelasticity are quite similar. Hence, for these examples the influence of the derivative of the seepage velocity can be neglected.

APPENDIX A: FUNDAMENTAL SOLUTIONS

The explicit expressions of the poroelastodynamic fundamental solutions for the simplified poroelastic model are given in the following for a 2-d and 3-d continuum.

A.1. Solutions in 3-d

The elements of the matrix $\mathbf{G}$ (19) are the displacements caused by a Dirac force in the solid

$$\hat{U}_{ij}^{\mathrm{s}} = \frac{1}{4\pi r \varrho s^2}\left[R_1 \frac{\lambda_4^2-\lambda_2^2}{\lambda_1^2-\lambda_2^2}\mathrm{e}^{-\lambda_1 r} - R_2 \frac{\lambda_4^2-\lambda_1^2}{\lambda_1^2-\lambda_2^2}\mathrm{e}^{-\lambda_2 r} + (\delta_{ij}\lambda_3^2 - R_3)\mathrm{e}^{-\lambda_3 r}\right] \tag{A1a}$$

with

$$R_k = \frac{3r_{,i}r_{,j}-\delta_{ij}}{r^2} + \lambda_k \frac{3r_{,i}r_{,j}-\delta_{ij}}{r} + \lambda_k^2 r_{,i}r_{,j} \quad \text{and} \quad \lambda_4^2 = \frac{s^2\varrho}{K+\frac{4}{3}G}$$

The pressure caused by the same load is

$$\hat{P}_j^{\mathrm{s}} = \frac{\alpha r_{,i}}{4\pi r \kappa (K+\frac{4}{3}G)(\lambda_1^2-\lambda_2^2)}\left[\left(\lambda_1+\frac{1}{r}\right)\mathrm{e}^{-\lambda_1 r} - \left(\lambda_2+\frac{1}{r}\right)\mathrm{e}^{-\lambda_2 r}\right] \tag{A1b}$$

For a Dirac source in the fluid the respective displacement solution is

$$\hat{U}_i^{\mathrm{f}} = \left(1-\frac{s\varrho_{\mathrm{f}}\kappa}{\alpha}\right)s\hat{P}_i^{\mathrm{s}} \tag{A1c}$$

and the pressure

$$\hat{P}^{\mathrm{f}} = \frac{1}{4\pi r\kappa(\lambda_1^2-\lambda_2^2)}[(\lambda_1^2-\lambda_4^2)\mathrm{e}^{-\lambda_1 r} - (\lambda_2^2-\lambda_4^2)\mathrm{e}^{-\lambda_2 r}] \tag{A1d}$$

In the above given solutions the roots $\lambda_i,\ i=1,2,3$ from (14) are used.

In the derivation of the poroelastodynamic boundary integral equation (21) several abbreviations (22) corresponding to an 'adjoint' traction or flux are introduced. First, the 'adjoint' traction solution is presented. However, due to the extensive expression only parts are given

$$\hat{T}^{\rm s}_{ij} = \left[((K - \tfrac{2}{3}G)\hat{U}^{\rm s}_{kj,k} + \alpha s \hat{P}^{\rm s}_{j})\delta_{i\ell} + G(\hat{U}^{\rm s}_{ij,\ell} + \hat{U}^{\rm s}_{\ell j,i}) \right] n_\ell \tag{A2a}$$

$$\hat{U}^{\rm s}_{kj,k}\delta_{i\ell}n_\ell = \frac{r_{,j}n_i}{4\pi r s^2 \varrho(\lambda_1^2 - \lambda_2^2)} \left[{\rm e}^{-\lambda_1 r}\left(\frac{1}{r} + \lambda_1\right)\lambda_1^2(\lambda_2^2 - \lambda_4^2) \right.$$

$$\left. - {\rm e}^{-\lambda_2 r}\left(\frac{1}{r} + \lambda_2\right)\lambda_2^2(\lambda_1^2 - \lambda_4^2) \right]$$

$$(\hat{U}^{\rm s}_{ij,\ell} + \hat{U}^{\rm s}_{\ell j,i})n_\ell = \frac{1}{4\pi r s^2 \varrho}\left[\frac{R_5 6}{r^3}\left(\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}\,{\rm e}^{-\lambda_1 r} - \frac{\lambda_4^2 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2}\,{\rm e}^{-\lambda_2 r} - {\rm e}^{-\lambda_3 r} \right) \right.$$

$$+ \frac{R_5 6}{r^2}\left(\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}\,\lambda_1 {\rm e}^{-\lambda_1 r} - \frac{\lambda_4^2 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2}\,\lambda_2 {\rm e}^{-\lambda_2 r} - \lambda_3 {\rm e}^{-\lambda_3 r} \right)$$

$$+ \frac{R_6 2}{r}\left(\frac{\lambda_4^2 - \lambda_2^2 k}{\lambda_1^2 - \lambda_2^2}\,\lambda_1^2 {\rm e}^{-\lambda_1 r} - \frac{\lambda_4^2 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2}\,\lambda_2^2 {\rm e}^{-\lambda_2 r} - \lambda_3^2 {\rm e}^{-\lambda_3 r} \right)$$

$$- 2r_{,n}r_{,i}r_{,j}\left(\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2}\,\lambda_1^3 {\rm e}^{-\lambda_1 r} - \frac{\lambda_4^2 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2}\,\lambda_2^3 {\rm e}^{-\lambda_2 r} - \lambda_3^3\,{\rm e}^{-\lambda_3 r} \right)$$

$$\left. - \lambda_3^2(\delta_{ij}r_{,n} + r_{,i}n_j\left(\lambda_3 + \frac{1}{r}\right){\rm e}^{-\lambda_3 r} \right]$$

with $R_5 = r_{,j}n_i + r_{,i}n_j + r_{,n}(\delta_{ij} - 5r_{,i}r_{,j}$ and $R_6 = r_{,j}n_i + r_{,i}n_j + r_{,n}(\delta_{ij} - 6r_{,i}r_{,j})$. The other explicit expressions are

$$\hat{Q}^{\rm s}_j = \frac{\alpha n_i}{4\pi r(K + \frac{4}{3}G)(\lambda_1^2 - \lambda_2^2)}[R_2 {\rm e}^{-\lambda_2 r} - R_1 {\rm e}^{-\lambda_1 r}] \tag{A2b}$$

$$\hat{T}^{\rm f}_i = \frac{1}{4\pi r\kappa(\lambda_1^2 - \lambda_2^2)}\left[\frac{n_j s(\alpha - s\varrho_{\rm f}\kappa)2G}{K + \frac{4}{3}G}(R_2 {\rm e}^{-\lambda_2 r} - R_1 {\rm e}^{-\lambda_1 r}) \right.$$

$$+ n_i {\rm e}^{-\lambda_2 r}\left(\frac{s(\alpha - s\varrho_{\rm f}\kappa)(K - \frac{2}{3}G)}{K + \frac{4}{3}G}\,\lambda_2^2 - \alpha s(\lambda_2^2 - \lambda_4^2) \right)$$

$$\left. - n_i {\rm e}^{-\lambda_1 r}\left(\frac{s(\alpha - s\varrho_{\rm f}\kappa)(K - \frac{2}{3}G)}{K + \frac{4}{3}G}\,\lambda_1^2 - \alpha s(\lambda_1^2 - \lambda_4^2) \right) \right] \tag{A2c}$$

$$\hat{Q}^{\mathrm{f}} = \frac{r_{,n}}{4\pi r(\lambda_1^2 - \lambda_2^2)} \left[\left(\lambda_2 + \frac{1}{r}\right)(\lambda_2^2 - \lambda_4^2)\mathrm{e}^{-\lambda_2 r} - \left(\lambda_1 + \frac{1}{r}\right)(\lambda_1^2 - \lambda_4^2)\mathrm{e}^{-\lambda_1 r} \right] \tag{A2d}$$

A.2. Solutions in 2-d

In 2-d, the expressions for displacements induced with a force in the solid are

$$\hat{U}_{ij}^{\mathrm{s}} = \frac{1}{2\pi s^2 \rho} \left(\frac{\lambda_4^2 - \lambda_2^2}{\lambda_1^2 - \lambda_2^2} R_1 - \frac{\lambda_4^2 - \lambda_1^2}{\lambda_1^2 - \lambda_2^2} R_2 + (\delta_{ij}\lambda_3^2 K_0(\lambda_3 r) - R_3) \right) \tag{A3a}$$

and the pressure for the same load is

$$\hat{P}_j^{\mathrm{s}} = \frac{\alpha r_{,i}}{2\pi\kappa(K + \frac{4}{3}G)} \left(\frac{\lambda_1 K_1(\lambda_1 r)}{(\lambda_1^2 - \lambda_2^2)} + \frac{\lambda_2 K_1(\lambda_2 r)}{(\lambda_2^2 - \lambda_1^2)} \right) \tag{A3b}$$

The displacement fundamental solution for a fluid source is

$$\hat{U}_i^{\mathrm{f}} = \left(1 - \frac{s\varrho_{\mathrm{f}}\kappa}{\alpha}\right) s\hat{P}_j^{\mathrm{s}} \tag{A3c}$$

and the pressure solution

$$\hat{P}^{\mathrm{f}} = \frac{1}{2\pi\kappa(\lambda_1^2 - \lambda_2^2)} (K_0(\lambda_1 r)(\lambda_1^2 - \lambda_4^2) - K_0(\lambda_2 r)(\lambda_2^2 - \lambda_4^2)) \tag{A3d}$$

with $R_k = (2r_{,i}r_{,j} - \delta_{ij})(\lambda_k/r)K_1(\lambda_k r) + r_{,i}r_{,j}\lambda_k^2 K_0(\lambda_k r)$ and $\lambda_4^2 = (s^2\rho)/(K + \frac{4}{3}G)$. K_0 and K_1 denote the modified Bessel Functions of the second kind.

In 2-d, the expressions for 'adjoint' traction and flux are

$$\hat{T}_{ij}^{\mathrm{s}} = [((K - \tfrac{2}{3}G)\hat{U}_{kj,k}^{\mathrm{s}} + \alpha s\hat{P}_j^{\mathrm{s}})\delta_{i\ell} + G(\hat{U}_{ij,\ell}^{\mathrm{s}} + \hat{U}_{\ell j,i}^{\mathrm{s}})]n_\ell \tag{A4a}$$

$$\hat{U}_{kj,k}^{\mathrm{s}}\delta_{i\ell}n_\ell = \frac{r_{,j}n_i}{2\pi s^2\varrho(\lambda_1^2 - \lambda_2^2)} [\lambda_1^3 K_1(\lambda_1 r)(\lambda_2^2 - \lambda_4^2) - \lambda_2^3 K_1(\lambda_2 r)(\lambda_1^2 - \lambda_4^2)]$$

$$\begin{aligned}(\hat{U}_{ij,\ell}^{\mathrm{s}} + \hat{U}_{\ell j,i}^{\mathrm{s}})n_\ell = \frac{1}{\pi} \Bigg[& \frac{\lambda_4^2 - \lambda_2^2}{\lambda_3^2(\lambda_1^2 - \lambda_2^2)} \left(R_7 \frac{\lambda_1}{r} \left(\lambda_1 K_0(\lambda_1 r) + \frac{2K_1(\lambda_1 r)}{r} \right) - r_{,i}r_{,j}r_{,n}\lambda_1^3 K_1(\lambda_1 r) \right) \\ & - \frac{\lambda_4^2 - \lambda_1^2}{\lambda_3^2(\lambda_1^2 - \lambda_2^2)} \left(R_7 \frac{\lambda_2}{r} \left(\lambda_2 K_0(\lambda_2 r) + \frac{2K_1(\lambda_2 r)}{r} \right) - r_{,i}r_{,j}r_{,n}\lambda_2^3 K_1(\lambda_2 r) \right) \\ & - \frac{R_7}{\lambda_3 r} \left(\lambda_3 K_0(\lambda_3 r) + \frac{2K_1(\lambda_3 r)}{r} \right) - \frac{r_{,n}(\delta_{ij} - 2r_{,i}r_{,j}) + r_{,i}n_{,j}}{2} \lambda_3 K_1(\lambda_3 r) \Bigg]\end{aligned}$$

with $R_7 = [r_{,n}(\delta_{ij} - 4r_{,i}r_{,j}) + r_{,j}n_{,i} + r_{,i}n_{,}](1/r)\lambda_1[\lambda_1 K_0(\lambda_1 r) + (2K_1(\lambda_1 r))/r] - r_{,i}r_{,j}r_{,n}\lambda_1^3 K_1$ $(\lambda_1 r)$.

The other explicit expressions are:

$$\hat{Q}^{\mathrm{s}}_j = \frac{1}{2\pi s^2 \rho}\left[\frac{r_{,j}r_{,n}}{\lambda_1^2-\lambda_2^2}(\alpha\lambda_4^2\lambda_2^2 K_0(\lambda_2 r) - \alpha\lambda_4^2\lambda_1^2 K_0(\lambda_1 r))\right.$$

$$\left. + \frac{2r_{,n}r_{,j}-n_j}{r(\lambda_1^2-\lambda_2^2)}(\alpha\lambda_4^2\lambda_2^2 K_1(\lambda_2 r) - \alpha\lambda_4^2\lambda_1^2 K_1(\lambda_1 r))\right] \tag{A4b}$$

$$\hat{T}^{\mathrm{f}}_i = \frac{1}{2\pi\kappa(\lambda_1^2-\lambda_2^2)(K+\frac{4}{3}G)}\left[2r_{,i}r_{,n}s(\alpha - s\varrho_{\mathrm{f}}\kappa)G\right.$$

$$\times\left(\left(\lambda_2^2 K_0(\lambda_2 r) + \frac{\lambda_2}{r}K_1(\lambda_2 r)\right) - \left(\lambda_1^2 K_0(\lambda_1 r) + \frac{\lambda_1}{r}K_1(\lambda_1 r)\right)\right)$$

$$-2(n_i - r_{,i}r_{,n}s(\alpha - s\varrho_{\mathrm{f}}\kappa)G)\left(\frac{\lambda_2}{r}K_1(\lambda_2 r) - \frac{\lambda_1}{r}K_1(\lambda_1 r)\right)$$

$$+n_i\left[\left(K-\frac{2}{3}G\right)s(\alpha - s\varrho_{\mathrm{f}}\kappa)\lambda_2^2 - \alpha s\left(K+\frac{4}{3}G\right)(\lambda_2^2-\lambda_4^2)\right]K_0(\lambda_2 r)$$

$$\left. -n_i\left[\left(K-\frac{2}{3}G\right)s(\alpha - s\varrho_{\mathrm{f}}\kappa)\lambda_1^2 - \alpha s\left(K+\frac{4}{3}G\right)(\lambda_1^2-\lambda_4^2)\right]K_0(\lambda_1 r)\right] \tag{A4c}$$

$$\hat{Q}^{\mathrm{f}} = \frac{r_{,n}}{2\pi(\lambda_1^2-\lambda_2^2)}[(\lambda_2^2-\lambda_4^2)\lambda_2 K_1(\lambda_2 r) - (\lambda_1^2-\lambda_4^2)\lambda_1 K_1(\lambda_1 r)] \tag{A4d}$$

with $R_k = (2r_{,i}r_{,j} - \delta_{ij})\frac{\lambda_k}{r}K_1(\lambda_k r) + r_{,i}r_{,j}\lambda_k^2 K_0(\lambda_k r)$.

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