(11)

Interpolation in square grid DTM

F. LEBERL*)

ABSTRACT

The present study aims at an evaluation of the problem of interpolation in square grid Digital Terrain Models (DTMs). Automation of photogrammetry, and the special advantages of measuring and interpolating in a square grid DTM justify the expectation that sampling of the terrain surface along a regular point pattern is very promising.

Performance of a DTM depends on the terrain itself, on the measuring pattern and point density in digitizing the terrain surface, and on the method of interpolating a new point from the measurements. The study of the interrelations among these various factors is based on a numerical experiment, in comparing a range of 6 terrain models, different interpolation procedures and spacing of the square grid varying from 10 m up to 450 m. This extremely large range of sampling densities is motivated by the expectation, that DTMs be also applied in small scale automatic photogrammetry.

The report consists of a description of the different interpolation algorithms investigated, then of the treatment of the preparation of the numerical experiment and finally of the analysis of the obtained results. Comparison of the different interpolation algorithms leads to the conclusion, that "linear prediction", "moving averages" and "patchwise polynomial interpolation" provide the highest accuracy. Consideration of cost shows, however, that the method with "moving averages" is comparatively rather expensive, so that the other two remain as the most effective interpolation methods. It thereby was clearly demonstrated, that weighting of

*) Lecturer in the Photogrammetric Dept. (Paper received July, 1973)

the observed point is of crucial importance. It was found, that the average improvement of accuracy in the numerical experiment by using e.g. linear prediction, rather than linear interpolation, amounts to 20 - 30%, with a maximum of approx. 50%.
In a comparison of using the 2x2, 4x4, or 6x6 surrounding reference points for interpolation of a new point, it was concluded, that no gain might be expected by using more than the 4x4 reference points. On the other hand, use of 4x4 points tends to be slightly superior to the use of only the 4 closest reference points.
It finally could be shown, that a linear relation exists between accuracy of interpolation and sampling density. The slope of the linear regression equation is correlated with the terrain type. An attempt was made to identify a simple indicator for the terrain type, namely the "normalized standard deviation of terrain relief".

RESUME

La présente étude est orientée vers l'évaluation des résultats de l'interpolation dans un modèle numérique de terrain observé selon un quadrillage. L'automatisation de la photogrammétrie et les avantages qui proviennent de la mesure et de l'interpolation selon un réseau carré permettent de penser que la numérisation de la surface du terrain selon un tel maillage fournira des résultats satisfaisants.
L'exécution d'un semis de points dépend du terrain lui-même, de la forme et de la densité des mailles et de la méthode d'interpolation d'un point nouveau à partir des points mesurés. L'étude des relations qui existent parmi les divers facteurs se base sur une expérience numérique, couvrant un choix de six modèles de terrain, et utilisant différentes méthodes d'interpolation selon un maillage qui varie de 10 à 450 m.
Ce large choix des densités d'échantillonnage est dicté par la supposition que les modèles numeriques de terrain peuvent été également utilisés dans la photogrammétrie automatique à petite échelle.
L'étude consiste en une description des différents algorithmes d'interpolation étudiés, de la préparation de l'expérience numérique et finalement d'une analyse des résultats obtenus.
Une comparaison des différentes méthodes d'interpolation conduit à la conclusion que la "prédiction linéaire", la "moyenne flottante" et "l'interpolation polynomiale à maille" sont les plus précises.

La comparaison des coûts marque que la méthode de la "moyenne flottante" est relativement plus chère. Il a aussi été clairement démontré que le choix des poids, dont on affecte les points observés, est très important. On a également trouvé que l'amélioration de la précision, concernant les cas étudiés, est approximativement de 20 à 30% (avec un maximum de 50%) pour la méthode de la "prédiction linéaire", comparativement avec la simple interpolation linéaire.
La comparaison dans l'utilisation de points de référence proches pour l'interpolation (2x2, 4x4, 6x6) à montré l'inutilité d'aller au-delà de 4x4 points. D'autre part il apparaît que l'on obtient une précision légèrement supérieure avec 4x4 points qu'avec 2x2 points.
Finalement la conclusion est que la relation entre l'erreur d'interpolation et la densité des points mesurés est linéaire. La pente de l'équation de régression est fortement correlée avec le type de terrain. On a essayé d'identifier un indicateur simple caractérisant le type de terrain sous la forme d'une "erreur moyenne normalisée du relief du terrain".

1. INTRODUCTION

The concept of Digital Terrain Models (DTM) started about 1955. It was created at the Massachusetts Institute of Technology in an attempt to automize some phases of highway design [33/1] . From there it has found limited application for the specific problems of automatic computation of profiles, cross sections, and earth work for alternative highway routes and other civil engineering projects.

A DTM consists of two components namely:
a set of representative points of the surface of the terrain, stored in the memory of a computer, and algorithms to interpolate any new point of given planimetric location or to estimate other data [20].

1.1 Objectives

This present study is the evaluation of interpolation in square grid DTMs and aims at answering the following, largely interrelated, questions.

- What density of measurements is required to obtain a specified accuracy of terrain representation?
- What is the relation between terrain types and the performance of a specified DTM?

- How do the various methods of interpolation compare?
- What is the optimum number of points to be used for interpolation of a new point?
- How should the given points be weighted in the interpolation?
- How does the accuracy of interpolation vary as a function of the position of the new point within the pattern of the given points?

Apart from these questions on accuracy, the computation efforts to interpolate single points and lines (contours, profiles) should be studied.

1.2 Past Efforts

Most of what has been published so far on DTMs was based on their application to civil (highway) engineering thus limiting them to large scale work with its high density digitizing of the terrain surface [1] , [2].

In view of recent developments such as correlators, digitally controlled orthophoto production, automated contouring etc., and the trend towards increasing automation of photogrammetry and cartography, application of the DTM to other than large scale work becomes a realistic possibility [22] , [35] . Therefore an analysis of the metric performance of digital terrain representation should be extended to smaller scales and consequently to lower density digitizing of the terrain surface.

In this report, considerable attention will be paid to the comparison of interpolation methods. Among the problems of applying DTM, interpolation is certainly an aspect of minor importance. This is especially so in highway engineering applications. At the session of Commission IV, on "Photogrammetry and Highway Engineering", during the ISP conference at Ottawa in 1972, this even led one of the speakers to state that one should not investigate interpolation methods in DTMs, since the obvious one to be applied in practice is linear interpolation. This statement seems to have been rather intuitive, since quantitative information on interpolation in DTMs is rarely published in photogrammetric literature. In the preparation of the present study, only five small tables could be found, in the publications by SILAR [30] , VIITA [33/3] , by NAKAMURA [25] , BEYER [6] and by LAUER [18] . These results will be discussed later. To a rather limited extent these publications allow conclusions on the above questions.

1.3 Restriction to Square Grid DTM

DTM data can be procured along contours or profiles, on a regular grid, or along terrain break lines and points.

It is expected, that sampling of DTMs in a regular point pattern is the most promising application of automated photogrammetry. Compared to other sampling methods, data procurement can be automized (electronic correlator) fully for the the regular point DTM, with substantially higher reliability. In addition, the organisation of data and its interpolation is greatly simplified with this type of DTM.

For special applications, digitizing of manually procured contours or profiles might be a powerful alternative to the regular grid DTM, if the contours (or profiles) are required as a cartographic product anyway, and if the preparation of slope charts, computation of earth work etc. is only an additional purpose. Although a DTM from digitised contours has a number of drawbacks (cannot be satisfactory automized; continuous sampling is less accurate than point by point sampling and requires subsequent data compression, since part of the sampled data is superfluous; interpolation is awkward) there is the advantage of varying sampling density (mainly in the direction of the contour, to a lesser extent across this direction), if the terrain is irregular ("autoreductive" [32] . This was the reason for using this method in [21] , and led DELIGNY [33/5] and SILAR [30] to the suggestion that for civil engineering purposes, sampling along contours should be adhered to in irregular terrain.

This suggestion, however, seemed not to take into account that a similar, or even superior, adjustment of density is possible also in DTMs sampled in a square grid. A very effective method is the one of "progressive sampling" as proposed by MAKAROVIČ [24] . A less effective, but also simple method is to measure a square grid DTM in separate rectangular blocks of a size suitable for simplification of subsequent data handling, whereby a number of blocks might add up to a photogrammetric model [15] . An obvious possibility for adjusting the sampling density is now to select it anew for each block.
Consequently this eliminates the principal arguments for sampling along contours, except if they themselves are required for cartographic purposes, and if plotting has to be done manually.

Consideration of "terrain break lines" or points in addition to the given DTM might in many cases be desirable. The problem of how these additional measurements should be incorporated into data processing exists in DTMs observed along contours, profiles, or on regular grids. This problem has not yet been systematically studied. Although a number of solutions do exist [31] , [34], [40] , the cost/benefit ratio for the inclusion of terrain break lines is not yet established. It is therefore suggested to leave this problem for a separate investigation.

1.4 Scope and Organisation of Study

The performance of a DTM depends on the terrain itself, on the measuring pattern and point density in digitizing the terrain surface, and on the method of interpolating a new point from the measurements [25]. This is illustrated in figure 1.

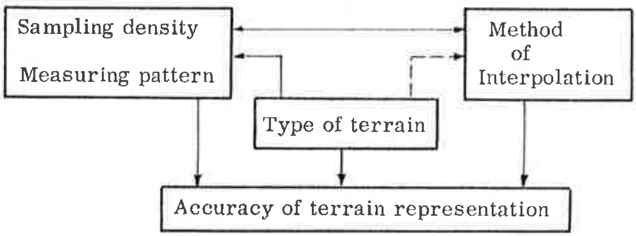

Figure 1.
Factors influencing the performance of a Digital Terrain Model

The method under study is a numerical experimentation. According to figure 1 an experiment to evaluate the performance of the DTM should encompass a range of terrain types, a number of measuring patterns and point densities, and interpolation procedures.

A number of interpolation algorithms have been programmed and tested in this study. Since some of them might not be generally known, they will be described in a separate chapter.
Next the preparation of the experiment will be treated, namely description of the measurements and of the six photogrammetric models on which the experiment was based.

Then, the results of the study are described and analyzed. The above list of questions will not be followed, but an attempt will be made to define an optimum interpolation method. Accuracy models will be given for the relation between terrain type, grid spacing, and interpolation error.

In the experiment, the highest accuracy was obtained when the surrounding 16 given points were used for the interpolation of a new point, applying a carefully selected linear prediction model or a patchwise or sliding polynomial interpolation. Overall optimization might, however, result in the use of a simple interpolation method, at the expense of accuracy, but with a reduction in computation time. This is, however, ultimately the task of overall optimization within the planning of an actual project. The intention of this paper can only be to provide information for this overall optimization and not the optimization itself.

2. INTERPOLATION METHODS

2.1 General

The problem of interpolation from discrete observations can be described as follows:
On a number of points P_i in n-dimensional space, called "reference space", vectors of dimension m are defined.
Interpolation consists of finding the unknown vectors to any number of other points P_k, using the known vector in points P_i.

As an interpolation problem, a DTM is simple.
The dimension n of the reference space is 2, since it consists of the XY coordinate plane. The dimension m of the vector to be found is 1, since the entities to be interpolated are the one-dimensional heights Z.

Interpolation and the related subject of transformation of inconsistent coordinates, are again in the limelight due to the recent re-definition of a number of traditional photogrammetric problems as interpolation problems and the breakthrough of the theory of stochastic processes in geodesy and photogrammetry. Examples are numerous in photogrammetric literature, and concern film deformation, lens distortion, model-, strip- and block deformations, etc. (ARTHUR [5] , SCHUT [27] , BEYER [6], KRAUS and MIKHAIL [16] , VLCEK [37] , BOSMAN et al. [9]).
Interpolation has also been studied intensively in other applied sciences such as geology (see [41]).

An attempt to classify the large number of possible interpolation procedures, to discuss them, and to make a choice for use in a particular problem is certainly difficult. This is why decisions on an interpolation procedure had and still have to be made on the basis of intuition, logical considerations, and experience. For the particular case of interpolation in a square grid DTM, an optimum procedure can, however, be defined, as will be shown in the discussion of the numerical results.

Interpolation of a one-dimensional random function which is defined on a 2 dimensional reference space is a problem of "surface-fitting". Three basically different approaches are possible, namely:

- interpolation by a single, global function;
- interpolation by piecewise, locally defined functions;
- pointwise interpolation.

In the first case, that of interpolation by a single function, all reference points are used simultaneously to define a single function $Z = f(x, y)$. For application to a DTM, this is usually inappropriate. Either the terrain is too irregular, so that the function cannot conform to it, or the number of unknowns to be determined is so large that the solution for the coefficients of the function is impossible or instable. This is so even in the case of relatively small areas. An effective single interpolation function is HARDY's "multiquadric", defined as a sum of second order surfaces [11] , [12] :

$$Z = \sum_{i=1}^{n} c_i \left[(X_i - X)^2 + (Y_i - Y)^2 + C_i \right]^{\frac{1}{2}} \qquad (1)$$

where the index i denotes the n reference points.
Although HARDY showed in various studies that this function is very flexible, especially if digitizing is done along terrain break lines, its application to DTMs would generally be very laborious: the computation of the unknown C_i from n measured reference points would require inversion of a n.n matrix.

Interpolation by piecewise functions involves dividing the whole area of the DTM into smaller patches and representing each patch by one chosen function. In this way, the problem of determining simultaneously a large number of unknowns can be overcome, although another problem is thereby created; viz. that along the boundaries of patches, "cracks" or discontinuities might occur [25] .

To avoid these, constraints can be introduced by forcing the individual functions in the patches to coincide along boundary lines, so that joint functions will result. This idea was used in [9] and [19] . If these joining conditions have to be introduced explicitly into the computation of the unknown function parameter, then again the problem of the simultaneous determination of a large number of unknowns follows, just as was the case for interpolation with a single function.

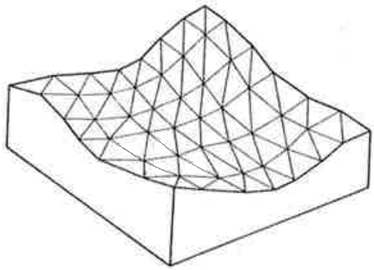

Fig. 2. Approximation of a terrain surface by plane triangles - linear interpolation LI

Therefore, functions have to be found for the individual patches so that the explicit consideration of the joining conditions is not necessary. A very simple example is the piecewise linear function or polyhedron (fig. 2). The reference points are used to define triangular plane pieces. In this case function values along the boundaries are automatically identical. More complex possibilities, using higher order polynomials also exist. LINKWITZ reports in [20] about a socalled "Mesh-surface" of Prof. COONS of MIT.*) Another example is the method by JANCAITIS and JUNKINS [13] . The main advantage of this approach is that the interpolation of a larger number of points per mesh does not cost very much more effort than the interpolation of a single point. In addition, efforts for contouring, profiling, etc., simplify with this method as compared to pointwise interpolation.

Pointwise interpolation avoids problems of computer storage, since each new point is interpolated independently, using only the surrounding subset of reference points. The coefficients of the interpolation function will vary from point to point. This increases flexibility although more computation is involved.

*) The MIT mesh-surface is also described in [42] .

Most of the existing operational DTMs are based on pointwise interpolation. The procedure is as follows: those measured points are selected in the DTM which fall inside a "critical circle (or square)" around the new point. These points are then applied to compute a weighted mean, low order polynomial or other function. The time-consuming process of defining the points inside the critical circle or square, is greatly simplified if the measurements are made in a regular point grid.

The piecewise linear interpolation, as illustrated is fig. 2, can also be defined as pointwise interpolation with three points, if a square or similar grid was measured. Similarly there are also cases where point- and patchwise interpolation is identical for square grid sampling. This will be discussed in the following section.

2.2 Description of Interpolation Methods Investigated

Due to its flexibility and local definition pointwise interpolation should provide results at least as accurate or even better than piecewise or single functions. This, and the numerical simplicity, is the reason that basically only pointwise interpolation is considered in the present study.

In the square grid DTM, pointwise and patchwise interpolation is identical, if the size of the patch in which a function is defined, is equal to the size of the meshes in the measurements. The results obtained for the pointwise methods are therefore also applicable to the specific case of patchwise interpolation.

For the case of more than the 4 closest reference points, four effective point/ patchwise interpolation algorithms have been selected for detailed study, namely:

- weighted mean,
- moving averages,
- linear prediction,
- minimum sized polynomial patches.

For the cases of 4 reference points, one obtains interpolation in a grid mesh by using the four corner points. Five algorithms have been studied:

- weighted mean,
- linear prediction,
- bilinear polynomial,
- two versions of linear interpolation.

This selection of interpolation methods is of course incomplete.
In theory, a large number of alternatives is available: e.g. orthogonal polynomials, Fourier series, etc. [19] . The methods which are selected for the present study seem, however, to be the most economic and powerful for application to DTMs. At this point, this is an intuitive statement.

A comparison of interpolation methods will, however, show, that carefully applied linear prediction provides optimum results, as good as those obtainable by interpolation with minimum sized patches and "moving averages". Theory of random functions proves that linear prediction is the optimum interpolation method, provided that certain conditions hold concerning the data. The fact that all three of these methods perform equally well is an indication that there would be no point in considering other alternatives.

2.2.1. Interpolation With Weighted Arithmetic Mean (AM)

A new height Zp is interpolated from the n surrounding reference values by:

$$Zp = \sum_{i=1}^{n} w_i \cdot Z_i \ / \sum_{i=1}^{n} w_i = \underline{w} \cdot \underline{Z}^t / \underline{w} \cdot \underline{u}^t \quad \ldots\ldots\ldots\ldots \quad (2)$$

The underlined variables represent matrices (or vectors). Thus:

$$\underline{w} = (w_1, w_2, \ \ldots\ldots \ w_n)$$

Vector $\underline{u}$ represents the unit vector.

The weights w_i should be a function of the distance d_i between the new point P and reference point i. In this interpolation method the following weight function will be used [37] :

$$w_i = 1 \ / \ d_i^k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Obviously, a large value of k increases the effect of the closest reference point(s) while reducing the influence of all other points.

The use of this method for application to DTMs has been advocated by LAUER in [18] .

2.2.2 Interpolation with Moving Averages (MA)

The surrounding n reference points are used to define the coefficients of a polynomial of order m. If the new point is chosen as the origin of the coordinates, then only the absolute term of the polynomial has to be computed. The name "Moving Averages" of order m, involving n points, is given to this method in the theory of stochastic processes, where it is used as a smoothing and interpolation procedure (YAGLOM [39]).
The name is based on the fact that the function value of a polynomial can always be expressed as a linear combination ("average") of the values in the reference points.

This method can also make use of weights. The polynomial is then computed by giving the reference points a lower weight with increasing distance from the new point, just as in the method of the arithmetic mean. In the present study, m is chosen to be 3 since higher order polynomials usually do not improve the interpolation [30] . Two weight functions are considered: firstly the one given in formula (3), and secondly the Gaussian curve:

$$w_i = \exp\left(-\frac{d^2}{k^2}\right) \qquad (4)$$

The use of this function for moving averages was proposed by ARTHUR [5] .
This interpolation method is for example used in the Czechoslovakian DTM [29] and in the French "semis de points" [33/5] , for "external block adjustment" by SCHUT [27] and in the contouring programme of CALCOMP [3].
For the case of a square grid DTM, a moving average with constant weight w = 1 is identical to patchwise polynomial interpolation. The patches are the minimum square meshes. Inside a single mesh the coefficients of the moving average with w = 1 do not change and so cracks may occur along the boundaries of adjacent squares. Cracks can only be avoided if the weight w is varied as a function of distance d. NAKAMURA [25] , however, showed that the order of magnitude of these cracks is negligibly small, about 10% of the interpolation error (i. e. the difference between interpolated and measured check point).

2.2.3. Interpolation by Linear Prediction or Least Squares Interpolation (LP)

The correlation of heights in the terrain must be represented by a correlation function. Any pair of heights is thus correlated according to a function of the distance, and eventually direction between or location of the two points.

If n reference points are to be applied in the interpolation, a "trend function" (due to numerical simplicity a polynomial t (x, y) of order m) is first defined by the n points. The residuals only are then used for actual interpolation. The polynomial is called "global trend", and is required to give the reference values statistical homogeneity*). Under such conditions the Linear Prediction method is optimal among all linear interpolation methods. Isotropy of the data (correlation between 2 points depends only on distance, not on direction) is unnecessary for optimum results from linear prediction. If the data are, however, anisotropic then this must be accounted for by introducing an anisotropic correlation function w (d, α), where α represents the direction between 2 points.

In the present application, only an isotropic correlation function w (d) is considered, since the computation of anisotropy is practically not feasible, the number of known points being too small. Using the given correlation function w (d), a covariance matrix ${}_nQ_n$ can be found for the n reference points, such that:

$$\underline{Q} = \begin{bmatrix} 1 & w(d_{12}) & w(d_{13}) \cdots & w(d_{nn}) \\ w(d_{21}) & 1 & w(d_{23}) \cdots & w(d_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ w(d_{n1}) & w(d_{n2}) & w(d_{n3}) \cdots & 1 \end{bmatrix}$$

where d_{ij} is the distance between the i^{th} and j^{th} reference point. A covariance vector ${}_1q_n$ can be defined between the new points and the reference points:

$$\underline{q} = [\, w(d_1) \,,\ w(d_2) \ \ldots\ldots \quad w(d_n) \,]$$

where d_i is the distance between the new and given point.

The interpolation formula is then:

$$Z_p = t\ (X_p\,,\ Y_p) + \underline{q} \cdot \underline{Q}^{-1} \cdot \Delta \underline{Z}^{t} \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

*) Reference values are inhomogeneous if their stochastic properties vary with the planimetric location.

where Δ Z is the vector of residuals in the reference points after subtraction of trend t.

It cannot be the purpose of the present study to given an outline of the theory of this interpolation procedure. Therefore for further details the reader is referred to the literature, e.g. [16] .

In the numerical investigation, the following correlation functions have been chosen from the literature [10] , [16] , [38] :

$$w(d) = c / (1 + d^2/k^2) \quad \ldots\ldots \quad (6a)$$

$$w(d) = c / \exp (d^2/k^2) \quad \ldots\ldots \quad (6b)$$

The order m of the trend polynomial is taken to be 0 and 2. The case m = 0 represents the simplest trend computation. On the other hand there is no point in increasing m beyond 2 or 3, since higher order polynomials usually do not improve interpolation [30] . Linear prediction has been advocated by KRAUS [35] , who is using it in a programme for automatic contouring. In [18] , LAUER investigated the method for application to DTMs sampled along contours. He concluded, that the benefits derived from the more complex linear prediction do not justify the extra costs, as compared to the weighted arithmetic mean.

It is of interest that, long before linear prediction entered the field of geodesy, ARTHUR proposed in 1957, and published in 1965 an interpolation method which is identical to linear prediction [4] :

$$Z_p = \underline{q} \cdot \underline{k}^T$$

$$\underline{k}^T = \underline{Q}^{-1} . \underline{\Delta Z}$$

where the elements of vector $\underline{k}$ are called "constants characteristic of the controls Δ Z". The covariance function w(d) is called the "attenuation function" and is of the shape:

$$w(d) = 1 - d / a$$

In a new contribution [5] , ARTHUR uses expression (4).

2.2.4. Polynomial Interpolation with Minimum Sized Patches (PMA)

Patchwise interpolation is a well known principle in cartography and geodesy, and is applied in the transformation of one cartographic projection into another. In order to avoid "cracks" between adjacent patches, a linear function must be used without overdetermination, such as in method LI, or POL. In the case of non-linear functions, joining conditions should possibly not be considered simultaneously in the computation of the function parameters. GOTTHARD[10] has shown an interesting solution for application to photogrammetric model connection, in which a non-linear function is used, without the occurrence of cracks in adjacent models.

A solution to a similar problem, but with another technique, is that proposed by JANCAIRIS and JUNKINS [13] .

The idea is the following: In the corner points of a square patch, tangents t_x and t_y in the direction of the sides of the square are computed, using a certain number of surrounding points. As a result one obtains two tangents in each corner point of the patch in addition to the known height. This gives 12 known values per patch (4 corner points x 3 values = 12).

A polynomial with 12 coefficients can be determined by these 12 values:

$$Z_p = q_0 + q_1X + q_2Y + q_3XY + q_4X^2 + q_5Y^2 + q_6 \cdot X^2 \cdot Y + q_7XY^2 + q_8 \cdot X^3 + q_9 \cdot Y^3 + q_{10}XY^3 + q_{11}X^3Y \qquad (7)$$

If this polynomial is computed from the heights and tangents in the corner points, then it will produce the same function values along the boundaries as the polynomials in the adjacent patches.

The method described works with polynomials of 12 unknown coefficients. Solutions are also possible with e.g. 16 unknowns. For the determination of the 4 extra polynomial coefficients, 4 extra tangents are required, for example in diagonal direction (t_{xy}). Such a method has been developed and programmed by K. TEMPFLI at ITC. The increase of the number of unknowns to 16 produces not only continuity of the functions along the boundaries, but also ensures that the first derivatives in the direction normal to the boundary lines will be continuous.

The most accurate terrain representation is obtained, if the patches exactly coincide with the squares of the DTM. This has, however, the disadvantage that 3.n values have to be stored in a DTM of n points (if the original data are not kept in the memory). Enlarging the size of each patch so that it covers more than one square of the DTM saves computer memory at the expense of accuracy.

The definite advantage of this patchwise approach is the fact that the DTM is given as a continuous surface. Any following operation such as contouring, profiling, etc. can be carried out very effectively, contrary to pointwise interpolation, and the result will always be continuous.

In the numerical experiment, method PMA is applied in versions PMA1 and PMA2. In PMA1, the tangents in the corner point of a mesh are computed by means of weighted polynomials through the 16 points surrounding the centre of the mesh. In PMA2, the tangents in each direction of the network (t_x, t_y) are computed from the 3 closest reference points in the x- or y-direction respectively. For PMA1 and PMA2, the patch size equals the minimum square mesh. A polynomial according to equation (7) is used.

2.2.5. Interpolation with Bilinear Polynomial (POL)

If only 4 reference points are used for interpolation, then a bilinear polynomial of the form

$$Z = a_o + a_1x + a_2y + a_3xy \quad \ldots\ldots \quad (8)$$

is just defined by these points. In a regular square grid, no cracks will occur along the boundaries of meshes since the function then only depends on the two closest points, not on all four. This interpolation can be interpreted as a piecewise, or also as a pointwise procedure of interpolation. It has been advocated by BENNER and SCHULT in [7] .

2.2.6. Linear Interpolation (LI)

A new point is interpolated using the 3 closest reference points in:

$$Z_n = \tfrac{1}{2} (Z_1 + Z_3 + (Z_2 - Z_1). X + (Z_2 - Z_3). Y) \quad \ldots\ldots \quad (9)$$

In the case of a square grid, this is identical to surface fitting by plane triangles, where 3 points define one triangle, such that

$$Z_n = a_0 + a_1X + a_2Y \quad \ldots\ldots\ldots \quad (10)$$

This method is used in the Finnish DTM [33/3] and is also proposed by a number of authors, such as TERNRYD [35] , and BOSMAN [9] .

2.2.7. Double Linear Interpolation (DLI)

Another possibility still is, if a new point is found from two linear interpolations, in two triangles (see fig. 3), and the arithmetic mean of the two values is taken:

$$Z_n = .5 * (Z_{n1} + Z_{n2})$$
$$Z_{n1} = a_0 + a_1.X + a_2.Y \quad \ldots\ldots\ldots \quad (11)$$
$$Z_{n2} = b_0 + b_1.X + b_2Y$$

a_0, a_1, a_2 are defined by the reference points 1, 2, 3, and the coefficients b_0, b_1, b_2 by the reference points 1, 2, 4, whereby points 1 and 2 are the two closest reference points.

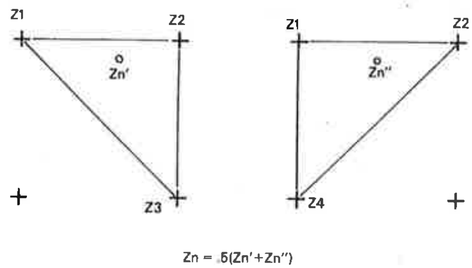

Figure 3. Double Linear Interpolation (DLI)

The immediate question arises why not a least squares fit of equation (11) to the 4 reference points? The answer is that in this case, cracks could occur along the boundaries of adjacent meshes. Method DLI was proposed by SCHATZ in [26] . It differs from POL only in the location of the reference points.

3. PREPARATION OF NUMERICAL EXPERIMENTS

The numerical experiments described in this report are based on measurements carried out for a related study [14] . The selected terrain-models, and the measurements in these models will now be described.

3.1 Terrain

Six photogrammetric models of different terrains were used as an input to the study. The problem one immediately faces when "different types" of terrain must be selected, is concisely and quantitatively to describe a terrain type. The most successful, although quantitative, method of describing a terrain type is probably by means of a contour map. Often photogrammetrists use only the maximum height difference in the terrain to describe it quantitatively. Geomorphologists, on the other hand, earn their living by the development and application of complicated systems of terrain classification.

An accepted and obvious quantitative terrain classification does not yet exist for the purpose of studying the effect of terrain shape on the results of various photogrammetric processes. But even if it did exist, then there still remains the problem that any large area of the Earth's surface is highly inhomogeneous *). Its properties can vary strongly from one part to another.

Category	description	t_r = t/hectare
I	regular, nearly plane surface elements	$t_r \leq 10$
II	regular, varying surface, oval shapes	$10 < t_r \leq 20$
III	irregular surface	$t_r > 20$
IV	artificial, man made surface	large number of artificial edges

Table 1.
SILAR's four terrain classes [30] .
t is the number of local extrema and/or terrain break lines.

*) Terrain (a random function) is called 'inhomogeneous', if its statistical properties vary with the planimetric location.

Attempts have been made to produce a terrain classification for photogrammetric purposes. SILAR [30] arrived at four terrain classes, simply by counting the local extrema and break lines of the terrain per unit area. The resulting scheme is given as table 1. Another possibility is that of interpreting the topography as a random function and describing it by means of Fourier series, or correlation functions. Numerical experiments prove, however, that the inhomogeneity of terrain prevents successful straightforward application of these concepts.

Name	Princ. distance c (mm)	Flying height (m)	Scale number	Photo-number
Wiesentheid	153.3	700	4 800	4 - 5
Welten	152.1	600	4 000	18 - 19
Kowloon	152.0	600	4 000	5769 - 5770
Oberschwaben	150.	4 200	28 000	655 - 656
South Wales	152.1	5 500	37 000	4763 - 4764
Surenen	152.1	3 750	25 000	3579 - 3580

Table 2.
Summary of data on the selected six terrain models

Six photogrammetric models, three at a scale of about 1 : 4000 and another three of about 1 : 30,000 were chosen so that conclusions could be reached on DTMs ranging from a high density of measurements, in a square grid of about 10 m side length, to a low density in square grids of 400 m sidelength. Table 2 summarizes the information on the six selected models.

Figure 4 shows small contour maps of parts of the six selected test models. An attempt to classify these terrain surfaces according to SILAR shows that the models "Wiesentheid" and "Oberschwaben" largely fall into terrain category I, "Welten" and "South Wales" into category II, and "Kowloon" and "Surenen" into category III.

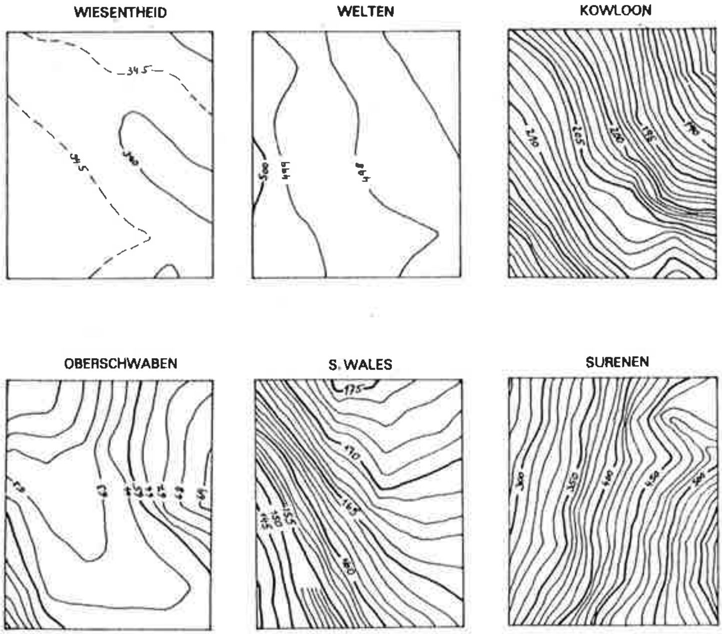

Fig. 4. Contour plots of sample areas of the 6 models used for the numerical experiment. The 3 upper plots cover an area of 60x50 m^2, the lower plots an area of 240x200 m^2.

Finally, an attempt was also made to use normalized correlation functions of terrain surfaces for classification.
The computational procedure as well as the physical meaning of correlation functions are well documented in photogrammetric literature, e.g. in [16].

Table 3 summarizes the coefficents of the computed correlation functions. The attempt to use them for classification has been unsuccessful.

Terrain model	cov = $c/(1+d^2/k^2)$		cov = c/eep(d/k)		σ_t	s	σ_n
	c	k	c	k	(m)	(km)	
Wiesentheid	.84	19	.84	28	.28	.2	1.4
Welten	.80	12	.80	50	1.06	.2	5.3
Kowloon	.98	14	.98	22	5.32	.2	26.6
Obersschwaben	.86	100	.86	300	3.97	.9	4.4
South Wales	.99	147	.99	235	12.3	.8	15.4
Surenen	.91	243	.91	430	32.0	1.0	32.0

Table 3.
Coefficients c, k of normalized correlation functions of the six selected terrain models. σ_t is the standard deviation of terrain relief, after trend subtraction. s is the side length of the squared terrain model, σ_n is the "normalized" standard deviation: $\sigma_n = \sigma_t/s$

The reason for this is, that the trend function*), the size of the area for which the function is computed, the inhomogeneity and anisotropy**), and even the density of measurements would have to be considered for classification. This leads to a system too complex to be useful for the present purpose.

3.2. Measurement of DTM

3.2.1. Pattern of Reference Points for Interpolation of New Points

A number of measured points ("reference points") must be used for the interpolation of a new point, which is defined by its planimetric X, Y coordinates. As has been stated previously, the regular point grid allows for a more economic procedure to select these reference points as compared to other measuring modes.

A new point is always situated in a square formed by those 4 measured points which are closest to the new point. The study considers the results by the use of only those 4 points in the interpolation. Increasing the number of points to more than 4 would be logical, if all those measurements which are within a "critical circle" around the new point are taken as reference points. The radius of this circle should be the maximum distance across which points could still contribute to the interpolation. Replacing the critical circle by a "critical square" simplifies the selection of reference points, because the necessity of time consuming computations of distances to all, or at least very many, points of the DTM is avoided.

In order to define the length of the side of the "critical square" not only the use of the surrounding 2 times 2 points, but also the 4 times 4 points, the 6 times 6 points, will have to be considered, or in general, the surrounding n points with

$$n = 4 . i^2 \qquad i = 1, 2, 3, \ldots \qquad \ldots\ldots \quad (12)$$

*) Corr. functions are computed for random functions with mean = 0. Therefore, residuals with respect to a "trend" are the input to compute correlations. Trend represents the mean. In the simplest case, it might be a mean horizontal plane. For table 3, it was a 2nd order polynomial.

**) Terrain is "anisotropic", if corr. between 2 points P1 P2 depends on the direction of the vector $\overline{P1\,P2}$.

In the present study, only three cases, with $i = 1, 2, 3$, are treated, which is sufficient to see how the interpolation depends on the number of reference points.

In order to also obtain an accuracy model describing the relation between interpolation and grid spacing, the test is not only carried out with the spacing of the original measurements, but also with three, five, and nine times this grid spacing factor ("g" = 1, 3, 5, 9). The actually tested cases are shown in figure 5. Vertically, interpolation is shown, with a fixed number of reference points, e.g. 4, but with a varying spacing g between the points. The multiples $g = 1, 3, 5$, and 9 of the originally measured grid spacing were chosen instead of another (such as 1, 2, 4, 8), because it allowed for a minimum number of measurements. For the cases with $g > 1$, not only the centre point, but also eccentric points are interpolated. Horizontally, then, the number of reference points is varied in figure 5, for a constant spacing of the grid.

3.2.2. Measurement of Experimental DTMs

The experimental DTMs, to be measured in the selected six photogrammetric stereomodels, should allow a test of the interpolation problems sketched in figure 5, in the ideal case in such a way that statistically sound results are obtained. The selection of the number of individual interpolations required to base sound conclusions on them is governed by two factors: first, the interpolation results in a square grid DTM are highly correlated and secondly, the terrain is rather inhomogeneous.

In order to reduce the effect of the correlations on the results, the sample measurements taken should be as large as possible. On the other hand, inhomogeneity of the terrain and limitation of resources for the study suggests that measurements should not be made in too large an area. In this trade off situation, an intuitive decision was taken.

A DTM of 35x35 points was measured. From these measurements, which are shown as crosses in figure 6, a number of 400 check points could be computed for every case shown in figure 5. These check points were also measured and are shown as dots in figure 6. The differences between the measured and computed values for the heights of these check points were used to draw the conclusions.

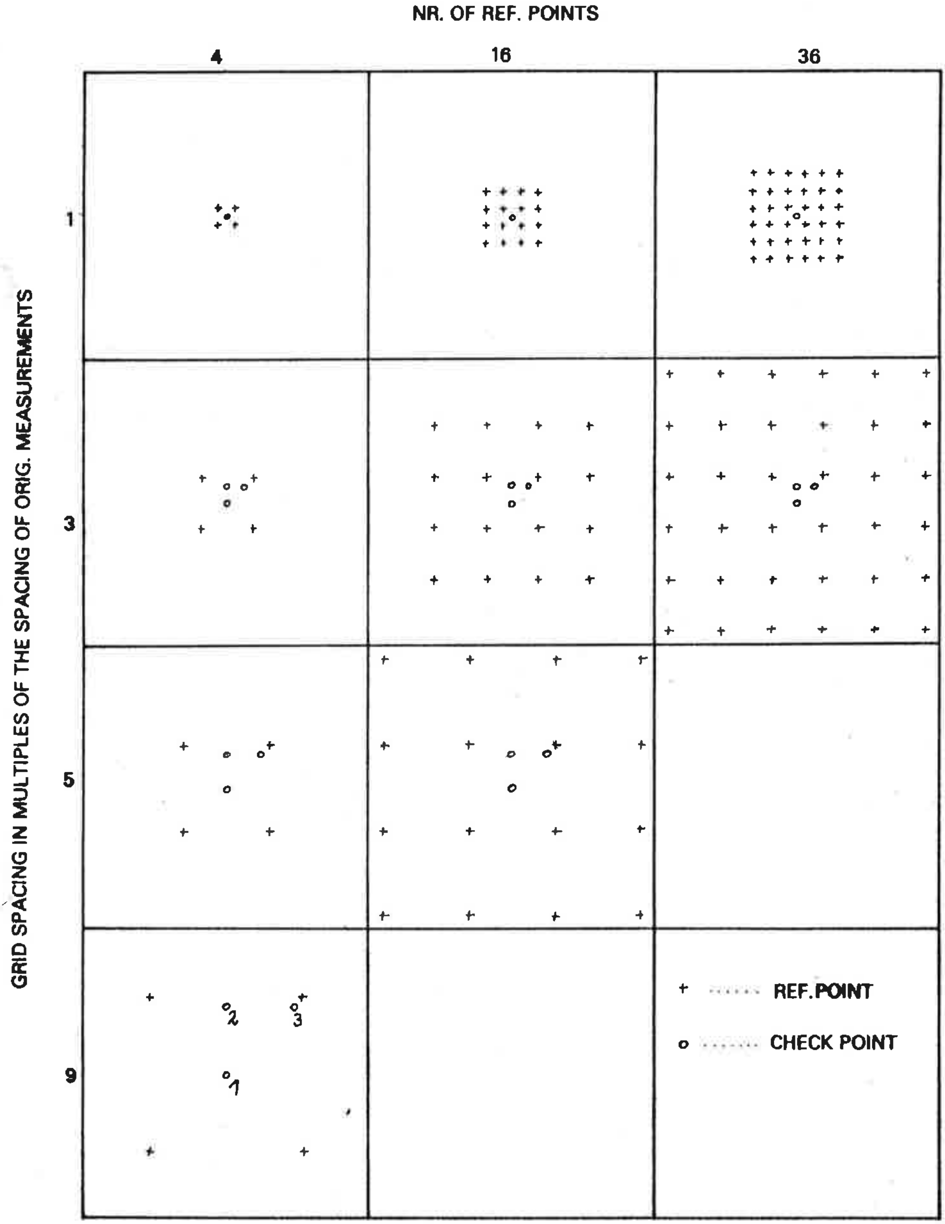

Fig. 5. Experimentally investigated interpolation cases, showing vertically the variation of grid spacing g = 1, 3, 5, 9 and horizontally the variation in the number of reference points 4, 16, 36

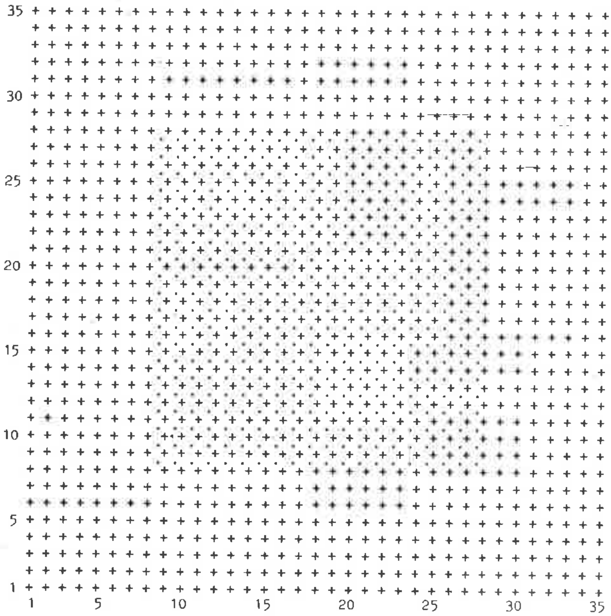

Figure 6. The observed square grid DTM

The 400 sample values are obtainable only if the centre points alone of the pattern in figure 5 are interpolated. However, when the eccentric points are computed as well, according to the same figure,the pattern of figure 5 only allows 256 sample values to be found.

The measurements were made on the AP/C Analytical Plotter, the output being on papertape. The spacing of the grid was chosen to be 10 m (2-4 mm at photo scale) for large scale photography, and 40-50 m (1-2 mm at photo scale) for small scale photography. The measuring accuracy can be estimated to be $\pm$ 0.01% to $\pm$ 0.02% of the flying height. The measurement consisted of automatic XY-movement, manual Z setting and triggering of recording.

4. RESULTS OF NUMERICAL EXPERIMENT

4.1 Introduction

Those interpolation procedures which were discussed in section 2 have been applied. Various weights were assumed, whenever weight was an input parameter to a method. The abbreviations used in the following sections are summarized in table 4.

Abbreviation	Explanation	Abbreviation	Explanation
LI	Linear Interpolation	LP	Linear Prediction
DLI	Double Linear Interpolation	g	Grid spacing
POL	Bilinear Polynomial Interpolation	σ_t^2	Variance of terrain relief
AM	Weighted Arithmetic Mean	σ_n	"Normalized" standard deviation of terrain relief
MA	Moving Average (Weighted)	e_1 e_2 e_3	RMS interpolation errors in three planimetric locations
PMA1	Patchwise Polynomial (Version 1)	e	$\sqrt{(e_1^2 + e_2^2 + e_3^2)/3}$
PMA2	Patchwise Polynomial (Version 2)	e'	e/e_{LI}

Table 4.
Summary of the abbreviations used in discussion of the results of the numerical experiment

The interpolation referred to the centers of gravity and two eccentric points, situated in patterns of 4, 16, and 36 reference points, in 6 terrain models, and with four different sampling densities. The total outcome of the experiment consists of 4212 interpolation cases.

For the highest sampling density, or minimum grid spacing, no eccentric points could be interpolated, but only the centres of gravity of the pattern of reference points. In this case of interpolating the centres of gravity and using only the four surrounding reference points, most interpolation methods produce the same result.

In each of the mentioned cases, at least 256 sample values are obtained for the difference v between a computed and measured terrain check point, so that the whole experiment totals more than a million interpolations.

Presentation of the results must necessarily be in a compressed form. In first instance, therefore, root mean square (rms) values are computed from the 256 sample values of each test case. The rms -value e_i is found from all computed v_i:

$$e_i = \sqrt{\frac{1}{256} \sum_{k=1}^{256} v_{ik}^2}$$

and is an estimate of the following expectation $\bar{e}_i$:

$$\bar{e}_i^2 = E \{ (v_i - E \{ v_i \})^2 \}$$

$$E \{ v_i \} = 0$$

$$\bar{e}_i^2 = E \{ v_i^2 \}$$

Here, v is the discrepancy between the measured and the interpolated heights. Index i refers to the planimetric location of the interpolated point. The discrepancies v_i are physically and algebraically correlated. This is due to the fact, that neighbouring check points are interpolated from partly the same reference points. The amount of correlation depends on the type of terrain, the method of interpolation, the pattern of reference points used for interpolation, and the location of the interpolated points. It is a complex problem to arrive at an estimate of this correlation, for a specific case of all those parameters which influence it. Therefore, the statistical properties of the e-values are not known. The values e_i do indicate orders of magnitude and interrelations amongst project parameters. They do not, however, relate to statistical confidence and standard deviation, due to the correlations among the original v_i.

Because six interrelated factors which influence the accuracy of a DTM were investigated and because each of these factors can vary over a large range of values, a concise description and discussion of the numerical results is hardly possible. Under the given circumstances it is considered a reasonable approach not to present the full range of computed e-values, but to make them available on request from ITC. The tables and figures presented in this report thus contain values derived from the original rms e_i- values. Each table is prepared with a specific item of discussion in mind.

For each interpolation case, check points in three planimetric locations were interpolated, namely the center of gravity and 2 eccentric points (figure 5). It is desirable to compress these 3 results to a single number to facilitate conclusions. A logical approach to this is to define a new r. m. s. interpolation error e as a function:

$$e^2 = \frac{1}{F} \int_{x=o}^{a} \int_{y=o}^{x} a_o + a_1 x + a_2 y \; dydx \quad \ldots\ldots\ldots\ldots \quad (13a)$$

The function to be integrated is a plane through the three points given by the coordinates $(0, 0, e_1^2)$, $(a, 0, e_2^2)$, (a, a, e_3^2). Index 1 refers to the centre of gravity, indices 2 and 3 to the eccentric points. The area F over which it is to be integrated is a triangle as shown in figure 7.
Consequently:

$$e^2 = \frac{1}{3} (e_1^2 + e_2^2 + e_3^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13b)$$

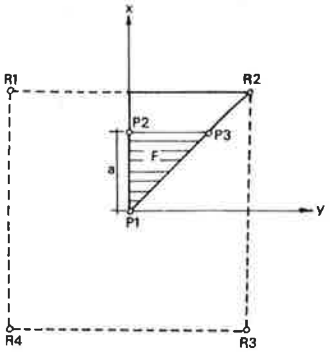

Fig. 7. Three new points P_1, P_2, P_3 are interpolated in the square formed by the ref. points R_1, R_2, R_3, R_4

Integration should actually extend over the whole triangle $\overline{P1, R2, H}$. But this would lead to unjustifiable extrapolation outside the triangle $\overline{P1, P2, P3}$. The integration over this smaller triangle produces e-values which are somewhat larger than those obtained from the triangle $\overline{P1, R2\ H}$.

For planning purposes it will often be the largest interpolation error occurring in the mesh, which is of highest importance. This is the error e_1 in point P1 (center of gravity). For volume computations, and for the study of a number of influencing factors (e.g. weight of reference points) the average e-value can be more useful.
In a number of cases during the following sections an attempt is made to compare the results obtained from different terrain and/or sampling densities. This comparison is facilitated, if the obtained e_i- or e-values are normalized by means of the e-value for linear interpolation LI:

$$e' = e \,/\, e_{LI} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

Values e' are then measures for the difference between the result after linear interpolation and the interpolation case in question. The normalized e'-values can be used to compute meaningful averages over different terrain and/or sampling densities.

The following discussion of numerical results will begin with the problem of weighting the reference points. Then, the interpolation methods are compared and an attempt is made to define the optimum number of reference points to be used for interpolation. The relation between grid spacing, terrain type, and interpolation error is treated next. Finally the variation of errors as a function of planimetric location is considered.

4.2 Optimum Choice of Weights and Other Parameters in Different Interpolation Methods

In a number of interpolation methods, each of the given reference points may obtain a certain weight in the interpolation, and/or other parameters may have to be chosen. Usually this is a matter of intuition. The numerical experiment allows for a few conclusions on the choice of weights and similar parameters.

Tables 5 - 9 have been prepared from the normalized e'-values, according to formula (14). Consequently they represent the difference between simple linear interpolation and the interpolation in question. The tables show only two grid spacings. This is justified by inspection of the full range of results which demonstrate that optimum weight is generally independent of grid spacing.

4.2.1. Arithmetic Mean

Table 5 illustrates that weighted arithmetic means from 4 reference points are superior to linear interpolation only, if in the weight function

$$w(d) = 1 / d^k \quad \ldots\ldots\ldots \quad (3)$$

k is larger or equal to 1. If more than 4 reference points were used in method AM, then it would even be necessary to choose $k \geqslant 2$ to approach the performance of method LI. This confirms the result obtained by LAUER in [18] .

No. of Ref. Points		4				16				36			
Weight		1	d^{-1}	d^{-2}	d^{-4}	1	d^{-1}	d^{-2}	d^{-4}	1	d^{-1}	d^{-2}	d^{-4}
g=30 m	Wiesentheid	1.40	1.07	1.07	1.07	2.20	1.67	1.33	1.13	3.00	2.33	1.60	1.13
	Welten	2.07	1.19	1.04	1.26	2.61	1.80	1.30	1.28	3.07	2.26	1.50	1.30
	Kowloon	1.45	.96	.81	.87	2.43	1.70	1.12	.91	3.35	2.36	1.37	.92
g=135 m	Oberschwaben	1.11	.88	.77	.75	1.73	1.38	1.02	.79	1.99	1.65	1.15	.80
	S. Wales	1.05	.87	.80	.83	1.78	1.40	1.05	.86	2.43	1.91	1.27	.87
	Surenen	1.46	.92	.84	.97	2.71	2.05	1.45	1.06	5.04	3.73	2.27	1.16

Table 5.

The effect of variation of weight on method AM. Shown are e'-values, computed according to formula (13b) and normalized according to formula (14), so that they represent the difference between method AM and LI:

$$e' = e_{AM}/e_{LI}$$

	Method	MA							PMA1		PMA2	
	Weight	1	d^{-1}	d^{-2}	d^{-4}	e^{-d^2}	e^{-2d^2}	e^{-4d^2}	d^{-1}	d^{-2}	d^{-4}	1
g=30 m	Wiesentheid	1.13	1.07	1.00	1.00	.93	.93	.93	.94	.94	.94	1.07
	Welten	.87	.85	.87	.87	.87	.87	.89	.89	.89	.89	.98
	Kowloon	.99	.77	.71	.69	.70	.68	.68	.71	.72	.70	.72
g=135 m	Oberschw.	.98	.78	.73	.70	.72	.69	.69	.72	.72	.70	.71
	S. Wales	.85	.76	.74	.75	.75	.75	.75	.77	.77	.77	.84
	Surenen	.89	.68	.62	.60	.61	.59	.59	.61	.63	.62	.83

Table 6.

The effect of variation of weight on method MA (a), and of the method of computing tangents in method PMA. The values shown are computed according to formulae (13b) and (14), from 16 reference points, and represent the difference between the investigated method and method LI: $e' = e/e_{LI}$

4.2.2. Moving Averages

Interpolations with DLI or POL do not provide a choice of weights. In moving averages, however, the introduction of weights can improve the results considerably (15 - 25 %). The left part of table 6 shows the relation between the accuracy and an increase of the power k in weight function (3) for the case of 16 reference points. This confirms that the selection of weight $1/d^3$ in the Swedish DTM is correct (NORDIN [33/2]). The right part of table 6 then demonstrates that it is even more effective to replace weight function (3) by (4):

$$w(d) = 1 / \exp(d^2/k^2) \quad \ldots\ldots\ldots\ldots \quad (4)$$

where d is given in multiples of grid spacing. Use of this function for moving averages has been proposed by ARTHUR [5] .

The best result, also for the case of 36 reference points, which is not shown in table 6, is obtained with $k \leq .7$. There is no point in choosing $k < .5$. The results would not improve, but the equation system to compute polynomial coefficients becomes singular. This confirms the results obtained by ARTHUR.

4.2.3. Patchwise Polynomial

For the patchwise polynomial interpolation, 2 alternatives (PMA1, PMA2) were studied. Table 6 shows the differences in performance between the 2 versions. The weight given to the reference points in version PMA1 is not critical. If the computation is, however, simplified to the extent of alternative PMA2, patchwise interpolation will be somewhat less effective. The differences amount to $\sim$ 10% on the average.

4.2.4. Linear Prediction

Two basic decisions have to be taken when applying linear prediction (LP). These concern the trend and the correlation function. In order to study the significance of "trend", two alternatives have been compared, namely:

a horizontal plane

a second order polynomial, each point with equal weight.

The comparison was only possible for cases with 16 and 36 reference points. No second order polynomial is defined by 4 points. Table 7 allows such a comparison. Clearly a horizontal plane is inappropriate. Results obtained with this trend are inferior to linear interpolation. It is only with a higher order trend function (polynomial of 2nd order), that the residuals sufficiently approximate the statistical properties of homogeneity and isotropy, so that linear prediction can fully be exploited.

As far as the correlation function is concerned, the following questions must be answered:

(i) What should be the coefficient c? If $c < 1$, then an uncorrelated component can be "filtered". If $c = 1$, the interpolation surface passes through the reference points.

(ii) Which correlation function is superior:

$$w(d) = c / (1 + (d^2/k^2) \quad \ldots\ldots\ldots\ldots \quad (6a)$$

$$w(d) = c / \exp(d^2/k^2) \quad \ldots\ldots\ldots\ldots \quad (6b)$$

(iii) Should the correlation function be constant for a certain grid spacing (and terrain), or should it vary for different grid spacings and terrain models?

Method	LP(1)		LP(2)	
No. of Ref Points	16	36	16	36
Welten (g=30 m)	.89	.85	1.24	1.33
S. Wales (g=135 m)	.76	.76	1.70	1.98

Table 7.
Linear Prediction (LP), using a horizontal plane as trend (LP(2)), and a second order polynomial (LP(1)). The given values are e' - values, obtained according to formulae (13b) and (14).

	Corr. Function	$1/(1+(d/k)^2)$				$.8/(1+(d/k)^2)$			
	k	.5	1	2	4	.5	1	2	4
g=30 m	Wiesentheid	1.07	1.07	1.00	1.00	1.15	1.09	1.00	1.00
	Welten	.93	.89	.89	.89	.95	.89	.89	.89
	Kowloon	.88	.81	.69	.66	.90	.85	.74	.67
g=135 m	Oberschwaben	.90	.85	.71	.67	.94	.89	.76	.75
	S. Wales	.81	.76	.75	.76	.84	.78	.75	.77
	Surenen	.85	.70	.61	.58	.88	.74	.65	.64

Table 8.
Linear Prediction (LP) with and without "filtering" (c = .8, c = 1). Shown are e' - values, obtained according to formulae (13b) and (14), from 16 reference points

	Corr. function	$1/(1+(d/k)^2)$				$1/e^{(d/k)^2}$			
	k	.5	2	5	10	.5	2	5	10
g=30 m	Wiesentheid	1.07	1.00	1.00	1.00	1.13	1.00	1.00	1.00
	Welten	.89	.89	.89	.89	.93	.91	.89	.91
	Kowloon	.84	.66	.66	.66	.93	.67	.67	.74
g=135 m	Oberschwaben	.88	.68	.67	.68	.97	.67	.68	.75
	S. Wales	.79	.76	.77	.76	.83	.78	.76	.81
	Surenen	.77	.58	.59	.58	.88	.60	.59	.67

Table 9.
Comparison of 2 correlation functions in method LP. Shown are e' - values obtained according to formulae (13b) and (14), from 16 reference points

(i) : Table 8 shows the normalized e'-values obtained from LP in all terrain models. The coefficient k of the weight function is varied horizontally. Table 8 shows the e'-values for coefficient c = 1 on the left, for c = 0. 8 to the right. Table 8 leads to the somewhat surprising result, that "filtering" does not improve the accuracy. Instead, the choice of c = 1 provides on the average a slightly (by a few percent) more accurate interpolation as compared to c = 0. 8. A similar result was also obtained by LAUER in [18] . The correlation function (6a) is used in table 8. Distances d are in units on the ground, so that correlation between neighbouring grid points reduces, if grid spacing changes from 10 to 30, 50m etc.

(ii) : Table 9 compares variants (6a) and (6b) of the correlation functions. The distance d is introduced in multiples of the grid spacing. The comparison shows that the two variants produce practically identical results.

A difference between the two results is that the one obtained from (6b) is more sensitive to a change of constant k than (6a). This is more obvious in the original data not included in this report where a larger range of k-values was considered. The fact that both functions produce the same optimum result, but (6a) over a larger range of k, makes this variant superior. The optimum result is obtained in most cases for $2 \leqslant k \leqslant 5$.

(iii): The difference between the 2 definitions of d (in meters on the ground, or in multiples of grid spacing) is not obvious from a comparison of tables 8 and 9, since both contain only e'-values for a specific grid spacing. However, from the full range of e'-values (over various grid spacings) it can be concluded that d should be introduced in multiples of the grid spacing. In this case, optimum results are always obtained for the same k. If d is expressed in absolute units in the terrain, then the optimum k increases with increasing grid spacing. (BEYER [6] also normalized the constant k by means of grid spacing).

A further conclusion is, that for all terrain models investigated, the same correlation function produced the best result. The optimum correlation function for LP is therefore variant (6a), with c = 1, $k \gtrsim 2$, and distance d normalized by means of the grid spacing.

So far, consideration of correlation was only based on "trial and error", or experimental optimization, as proposed by LAUER [18] . In theory, objective methods exist to compute the physical correlation of terrain heights. Table 3 showed the result of such a computation for the whole 20x20 check points of each terrain model.

In the actual interpolation, the correlation is required among residuals after subtraction of a local trend within 2x2, 4x4, or 6x6 reference points. According to KRAUS and MIKHAIL [15] , an objective computation of correlation can hardly be successful since the number of points on which it is based is rather small. An attempt has been made, however. Figure 8 shows, that the physical correlation found experimentally differs from the optimum found above by "trial and error".

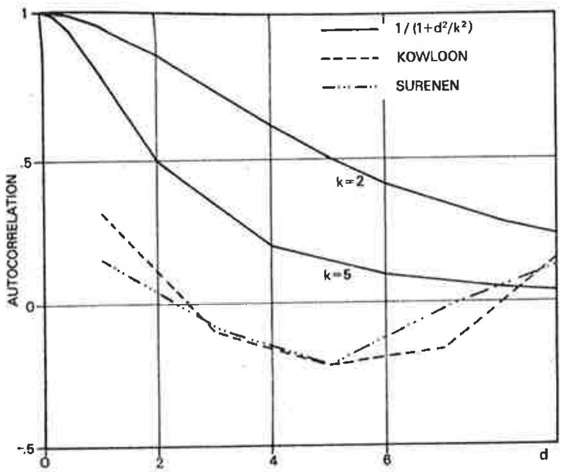

Fig. 8. Correlation functions for LP. The solid lines give the optimum function $1/(1+(d/k)^2)$, found by trial and error. The broken lines are the computed physical correlations. (d is in multiples of the grid spacing g = 1)

The most significant difference is the fact that such a large uncorrelated component seems to exist in the data. An attempt to "filter" this component by choosing $c < 1$ in the correlation function for LP failed, as can be seen in table 8. An explanation of this contradiction is only tentative at present and is based on the assumption that the theoretical functions are not defined well enough.

		e_1 (centre of gravity)					e_3 (eccentric)				
	Method	LI	POL	DLI	AM	LP	LI	POL	DLI	AM	
	Weight				d^{-2}	$\frac{1}{1+d^2/4}$				d^{-2}	$\frac{1}{1+d^2/4}$
g=30 m	Wiesenth.	.13	.17				.14	.14	.14	.14	.14
	Welten	.62	.61				.37	.37	.37	.46	.37
	Kowloon	4.41	3.08				2.19	2.08	1.92	2.02	2.01
g=135 m	Oberschw.	2.99	2.06				1.44	1.37	1.25	1.12	1.18
	S. Wales	5.34	3.86				2.53	2.39	2.17	2.11	2.22
	Surenen	12.7	8.43				5.42	5.05	4.54	6.24	5.39
	Average (LI = 100%)	1.00	.84				1.00	.96	.91	.99	.93

(a) 4 reference points

		e_1 (centre of gravity)				e_3 (eccentric)			
	Method	AM	MA	PMA1	LP	AM	MA	PMA1	LP
	Weight	d^{-4}	e^{-4d^2}	d^{-4}	$\frac{1}{1+d^2/4}$	d^{-4}	e^{-4d^2}	d^{-4}	$\frac{1}{1+d^2/4}$
g=30 m	Wiesenth.	.19	.16	.16	.16	.16	.13	.14	.13
	Welten	.64	.54	.54	.54	.81	.35	.34	.34
	Kowloon	3.35	2.56	2.62	2.51	2.88	1.82	1.80	1.77
g=135 m	Oberschw.	2.20	1.79	1.82	1.75	1.11	1.21	1.14	1.13
	S. Wales	4.10	3.45	3.45	3.47	2.20	2.15	2.04	2.06
	Surenen	9.82	6.11	6.23	6.04	9.78	3.96	3.83	3.77
	Average (LI = 100%)	.92	.73	.74	.73	1.14	.85	.84	.83

(b) 16 reference points

		e_1 (centre of gravity)			e_3 (eccentric)		
	Method	AM	MA	LP	AM	MA	LP
	Weight	d^{-4}	e^{-4d^2}	$\frac{1}{1+d^2/4}$	d^{-4}	e^{-4d^2}	$\frac{1}{1+d^2/4}$
g= 30 m	Wiesenth.	.19	.16	.16	.16	.13	.13
	Welten	.66	.54	.52	.81	.34	.34
	Kowloon	3.44	2.55	2.49	2.87	1.82	1.76
g=135 m	Oberschw.	2.23	1.79	1.73	1.11	1.22	1.12
	S. Wales	4.20	3.46	3.49	2.20	2.14	2.07
	Surenen	11.8	6.11	6.06	9.82	3.95	3.76
	Average (LI = 100%)	.96	.73	.72	1.35	.85	.82

(c) 36 reference points

Table 10.

Comparison of interpolation methods. Shown are a. m. s. e_i - values, in metres on the ground. In row "average", the average e_i' - values, after normalization with e_{LI}, are given, (the values of each row are divided by the value for LI, then the average is taken per column)

4.3. Comparison of Different Interpolation Methods

4.3.1. Accuracy

The accuracies of different interpolation methods are compared using the optimum weight and other parameters. The comparison is illustrated in table 10, showing the e_1 and e_3 values. The last row of each of the tables allows a comparison of the "average" performance of the different interpolation methods, using "average" e'_1 and e'_3-values ($e'_1 = e_1/e_{1\,LI}$), taken over the 6 terrain models at the indicated grid spacings. In addition, table 11 contains the average of the e'-values taken over all terrain models and all computed grid spacings

Method		LI	DLI	POL	AM	MA	PMA1	LP
Weight					$1/d^4$	e^{-4d^2}	d^{-4}	$1/(1+\frac{d^2}{4})$
No. of ref. points	4	1	.88	.89	.92			.88
	16				.97	.76	.76	.76
	36				1.03	.77		.76

Table 11.
Average performance of different interpolation methods. Shown are values obtained as average of the e' - values (formula 14) over all investigated terrain models and grid spacings

The following conclusions can be drawn for the tables 10 and 11:

- In general, the difference between interpolation methods is fairly small, i.e. on an average not more than 10 - 20 %. With 16 reference points interpolation is most accurate when using the methods of linear prediction (LP), moving averages (MA), or patchwise interpolation (PMA1).
- No significant difference exists between linear prediction, weighted moving averages and patchwise interpolation.
- For the case of 4 reference points, bilinear interpolation (POL), linear prediction (LP) and double linear interpolation (DLI) produce practically identical results, but on an average 10 - 15% better than linear interpolation (LI) and 10 - 15% inferior to MA, PMA1, or LP with 16 reference points.
- Interpolation with weighted arithmetic means (AM) is not an effective method. AM approaches the performance of linear interpolation only with the introduction of considerable weight differences.
- On an average, the differences between linear interpolation (LI) and other methods do not reduce with increasing grid spacing, nor when the terrain type changes (table 12). This conclusion is surprising. It was expected

that the differences between interpolation methods become smaller with increasing complexity (noise) of the terrain.
In the present experiment the largest improvement compared to linear interpolation was about 50% (terrain model "Surenen", method LP, 16 reference points) This percentage was also the largest discrepancy found by LAUER in [18] for the difference between methods LP and AM, however, for sampling along contours.

Terrain	σ_n	Grid spacing					
		10	30	45	50	135	225
Wiesentheld	1.4	(.90)	1.00		1.06		
Oberschw.	4.4			(.91)		.68	.68
Welten	5.3	(.96)	.85		.96		
S. Wales	15.4			(.85)		.76	.64
Kowloon	26.6	(.97)	.66		.67		
Surenen	32.0			(.91)		.58	.48

Table 12.
Effectiveness of linear prediction ($cov = 1/(1 + d^2/4)$) as compared to linear interpolation LI, versus grid spacing and terrain type. σ_n is "normalized standard deviation of terrain relief". Shown are the normalized e' - values, according to formulae (13b) and (14)

4.3.2. Costs

The comparison of costs for the different interpolation methods is based on the computation times. Such a comparison is subjective since it depends on the computer configuration and the programming. In the present case, the computer used was the PDP 11/45 of ITC. Programming was in Fortran IV, using the standard IBM subroutines for matrix operations.

There is no single answer to the problem of interpolation costs since these differ according to the number of points interpolated per square mesh. In this numerical experiment, only a small number of points were interpolated, namely 3 points per mesh.

Table 13 summarizes the overhead computation time per programme run, and the variable time per interpolation of a group of 3 new points per mesh. Excluded are the time for compilation and linking of the programme, as well as the time for in- and output. The data in table 13 thus only refer to the computation time.

Method	No. of Ref. points	Extra/Run (sec.)	Extra/Point (sec.)	Method	No. of Ref. points	Extra/Run	Extra/Point
DLI	4	.12	.04	PMA2	16	1.20	.10
POL	4	.12	.04	MA	16	2.17	.21
AM	4	.12	.04	MA	36	4.13	.43
AM	16	.12	.05	LP -	4	.12	.04
AM	36	.12	.08	LP	16	4.2	.12
PMA1	16	5.2	.25	LP	36	33.7	.23

Table 13.

Computation time for the interpolation methods under investigation. The indicated times exclude in- and output, and data transport, and were obtained on the PDP 11/45 of ITC.

Interpretation of this table leads to the following conclusions:

- Using 4 reference points for interpolation requires the same amount of computation time, irrespective of the method of interpolation. In the present case, this amounted to 25 points/second, with a negligible overhead.
- The fastest method of interpolation for more than 4 reference points is the weighted arithmetic mean (AM). Increase to 16 or 36 reference points hardly costs any extra computation time.
- Simplification of the patchwise interpolation from PMA1 to PMA2, thus simplifying the computation of tangents, represents a reduction of variable computation time by a factor 2.
- Comparison of methods MA and LP shows that LP is more economic. Variable cost is only half of that for the moving average. Fixed costs are considerably higher for LP, especially for 36 reference points, since inversion of a 36 x 36 matrix is required at the beginning. This, however, is easily compensated, if only about 150 points are interpolated.
- Computation time for method LP with 16 reference points is 4 times that of any of the simple algorithms using 4 reference points.
- Method PMA1, using 16 reference points, is 2 times more expensive than method LP with 16 points.

The fact, that the methods under study have been used in an experimental, rather than an operational DTM, causes no limitation to the validity of the variable cost of a single interpolation per mesh, except for the case of moving averages MA. There, the fact that the same pattern of new points was repeatedly interpolated (256 times per case), has been used to determine the inverse of the normal equation matrix only once, at the beginning of each run, so that this adds to the fixed costs. In actual applications, however, the planimetric location for which a new point is to be found within the square mesh, changes from mesh to mesh. Since also the weights change in this case, inversion of the normal equation matrix adds considerably to the variable costs. Full inversion is, however, not required. The new point is taken as origin of a local "moving" coordinate system, so that only the constant term of the polynomial has to be computed. Still, the relations shown in table 13 will be more unfavourable for MA, if operational application is considered. Reduction of costs, however, presumably at the expense of a loss of accuracy, is possible by chosing the order m of the polynomial in method MA ɔ be 1, rather than 3.

Interpolation of lines might require a larger number of points to be interpolated per mesh as compared to the case of a single point per mesh. In this case, method PMA1 can become more economic than method LP. Interpolation with PMA1, of a new point of which the 12 polynomial coefficients are known requires only 12 multiplications and additions. In method LP, this requires 17 * 16 multiplications and additions. For profiling, method PMA1 even allows for a very effective solution by simply intersecting the polynomial patch by a vertical plane.

4.4. Number of Reference Points per Interpolation

A number of interpolation algorithms for DTMs are based on only the 3 or 4 reference points closest to the new point to be interpolated. In this section an attempt is made to establish whether and how the accuracy of interpolation increases, if not only 3 or 4 but more reference points are used.

In section 3.2.1. it was stated, that the reference points applied in the interpolation should be within a "critical square" around the new point, so that the n closest reference points fall into this square of a side length of i grid spacings:

$$n = 4 . i^2 \qquad i = 1, 2, 3, \ldots \qquad (12)$$

In this study, only the cases i = 1, 2, 3 are considered. With formula (12), the question treated in this section can be reformulated:
Is the hypothesis correct that optimum results of DTM interpolation are obtained with i = 1 ?

		e_1 ... centre of gravity						e_3 ... eccentric point					
Grid spacing (m)		10			45			30			135		
No. of Ref. Points		4	16	36	4	16	36	4	16	36	4	16	36
LP	Wiesenth.	.10	.09	.09				.11	.13	.13			
	Oberschw.				.91	.83	.83				1.18	1.13	1.12
1	Welten	.26	.26	.26				.44	.34	.34			
	S. Wales				1.50	1.28	1.27				2.22	2.06	2.07
$1+\frac{d^2}{4}$	Kowloon	1.19	1.16	1.17				2.01	1.77	1.76			
	Surenen				2.92	2.65	2.66				6.30	3.77	3.77
MA (POL)	Wiesenth.	(.10)	.09	.09				(.14)	.13	.13			
	Oberschw.				(.91)	.84	.87				(1.37)	1.16	1.16
d^{-1}	Welten	(.26)	.25	.25				(.37)	.33	.34			
	S. Wales				(1.50)	1.33	1.39				(2.39)	2.05	2.06
	Kowloon	(1.19)	1.14	1.15				(2.08)	1.78	1.81			
	Surenen				(2.92)	2.68	2.77				(5.05)	3.77	3.83
AM	Wiesenth.	.10	.10	.10				.16	.16	.16			
	Oberschw.				.91	.94	.96				1.11	1.11	1.11
d^{-4}	Welten	.26	.26	.26				.80	.81	.81			
	S. Wales				1.50	1.59	1.63				2.19	2.20	2.20
	Kowloon	1.19	1.24	1.26				2.88	2.88	2.87			
	Surenen				2.92	3.10	3.18				9.67	9.78	9.82

Table 14.
Comparison of 4, 16, or 36 reference points. Values in metres on the ground. For the case of 4 reference points, method MA is substituted by method POL. The corresponding values are in brackets.

Table 14 compares the e_1, or e_3-values for the case of 4, 16, and 36 reference points (i= 1, 2, 3). This comparison is made for grid spacings of 10 m and 45 m for the e_1 value (centre of gravity), and grid spacings of 30 m and 135 m for the e_3- value (eccentric point in vicinity of reference point). The information of table 14 is also contained in table 10, however, in a less convenient way for the definition of the optimum number of reference points.

Table 14 also considers the method of interpolation itself since the optimum value for i might vary from one interpolation method to the other. Thus, in the results of methods AM (arithmetic mean), MA (moving average), and LP (linear prediction), as shown in table 14, a weight has been used which has been optimized by trial and error.

The following conclusions can be drawn from table 14.
With method MA and LP, interpolation in square grid DTM is more accurate if the 16 closest reference points are used, rather than only the 4 closest points (i.e. $i = 2$ is superior to $i = 1$). The improvement amounts to about 10%. The difference between use of 16 or 36 reference points is hardly significant, provided the weights are properly chosen. Therefore there is no point in chosing $i > 2$.

These conclusions apply also in the vicinity of a reference point (see right half of table 13, prepared from e_3-values). Interpolation with weighted arithmetic mean should, if at all, not be done with more than 4 points; the use of more than 4 points (16 or 36) does not improve, but rather impairs the results.

The conclusions, that 16 reference points do provide better interpolation results than 4 points, has also been found by LAUER in [18] , however, for sampling along contours.

.5. Grid Spacing, Type of Terrain, DTM-Accuracy

In order to establish models for the relation between accuracy, grid spacing and terrain type, both the "average" rms value e according to formula (13) and also e_1 (of only the centre of gravity of the reference pattern) are considered. Interpolation errors are largest in this centre of gravity and therefore of special importance for planning. Both the full range of available e- and e_1-values are shown in table 15.

To cover a maximum range of grid spacings the values of table 14 have to be obtained by an interpolation method using 4 reference points. With methods using 16 or 36 reference points, the full range of grid spacings was not obtained (see figure 5).

	Grid spacing (m)	10	30	45	50	90	135	225	405
e_1 - values (centre)	Wiesentheid	.10	.17		.25	.42			
	Oberschw.			.91			2.06	3.27	4.72
	Welten	.26	.61		1.10	1.72			
	S. Wales			1.50			3.86	6.19	10.5
	Kowloon	1.19	3.08		5.29	9.27			
	Surenen			2.92			8.43	17.4	41.0
e - values (average)	Wiesentheid	(.10)	.16		.21	.29			
	Oberschw.			(.91)			1.71	2.47	3.56
	Welten	(.26)	.54		.94	1.56			
	S. Wales			(1.50)			3.38	4.88	7.83
	Kowloon	(1.19)	2.60		4.17	7.26			
	Surenen			(2.92)			6.77	12.8	30.0

Table 15.

Grid spacing, terrain, interpolation errors (rms. e and e_1 values in m).
For grid spacings 10 and 45 m, no average e values, but only e_1 is known; the corresponding values are in brackets

Hence, table 15 was prepared with the results of method DLI. For e_1, these are identical with those from AM, POL, and LP.

4.5.1. Grid Spacing and DTM Accuracy

The most significant effect on the accuracy of digital terrain representation is, of course, produced by the spacing between the sampled terrain heights. Figures 9 illustrate graphically the relationship between sampling density and interpolation error. Graphs 9a, b show this relationship separately for each of the three interpolated points P1, P2, P3.

To stress the differences between these 3 points, a logarithmic scale has been chosen for the ordinate. Point 3 is located at a constant distance from the closest reference point, irrespective of the sampling density (see fig. 5). Consequently, the accuracy of interpolation is hardly affected in point 3 by a change of sampling density. For points 1 and 2, however, the distance from the closest reference point increases with decreasing sampling density.

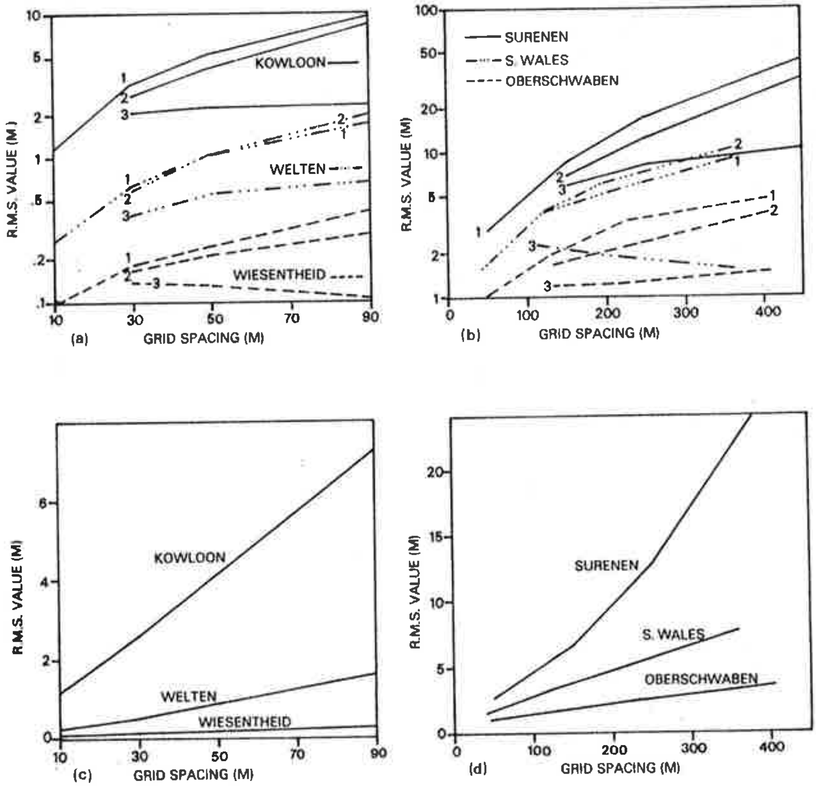

Fig. 9. Relation between accuracy and grid spacing of DTMs. (Note that a logarithmic ordinate has been used in figs (a) and (b)

Figures 9c,d illustrate the relationship between sampling density and average interpolation error e according to formula (13b). This relationship appears to be fairly linear, with a slight nonlinearity only for model "Surenen".

This suggests,for the investigated range of grid spacings and terrain types,a linear regression polynomial:

$$e = a_0 + a_1 x \qquad \text{(Surenen: } + a_2 x^2\text{)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

where x = grid spacing. Coefficient a_0 represents the measuring error. Table 16 shows the computed regression coefficients for the six models studied. The slope of the regression polynomial (15) varies from .02 (Wiesentheid) up to .77 (Surenen). The 2 models which fall into terrain class 1 according to SILAR [30] , have a slope $a_1 < .1$; terrain class 2 : $a_1 \approx .2$; terrain class 3 : $a_1 \approx .6$ to .8.

	a_0	a_1	σ_t(m)	σ_n
Wiesentheid	.1	.02	.28	1.4
Welten	.1	.16	1.06	5.3
Kowloon	.1	.61	5.32	26.6
Oberschw.	.6	.08	3.97	4.4
S. Wales	.7	.22	12.3	15.4
Surenen	.5	.78	32.0	32.0

Table 16.
Regression coefficients of formula (15) Values a_0 are in metres, and represent accuracy of measurements. σ_n is the "normalized standard deviation of terrain relief", and σ_t is the proper standard deviation of relief (see table 3)

Equation (15) confirms BEYER's result, obtained in the study of sampling for flat terrain [6] . BEYER found that the regression lines were linear and that their slopes varied between .0009 and .01 in 6 investigated models of flat terrain.

Further results on the relation between sampling density and grid spacing have been published by VIITA [33/3] , SILAR [30] and NAKAMURA [25] . Table 17 summarizes these results. The range of grid spacings, and the description of terrain types are, however, so limited that sound comparison with the present study is impossible.

Author	VIITA [33/3]		SILAR [30]			NAKAMURA [25]		
Method	LI		MA			PMA		
Grid spacing (m)	5	10	5	10	15	10	20	30
Accuracy (m)	.18	.23	.14	.19	.28	.54	1.17	2.38
Terrain	unknown		Category I			Category III?		

Table 17.

Interpolation results in regular point grid DTMs, published by VIITA, SILAR, and NAKAMURA. SILAR's MA did make use of weights. The patchwise moving average of NAKAMURA uses a full 3rd order polynomial, and 12 tangents. Continuity is, however, not achieved along boundaries

When the relation between sampling density and grid spacing is quantitatively described, the question arises whether an "indicator" for the slope a_1 of the basically linear relation (15) can be found. Such an indicator refers to the type of terrain. This will be discussed in the next section.

4.5.2. Terrain Type and DTM Accuracy

If a practical use of graphs 9 is considered, then these must, in the first instance, be combined with figures 3. On the basis of the contour plots, the terrain for which a DTM has to be established is classified, and the corresponding curve can be selected (or interpolated) in figure 9. This curve relates DTM accuracy with grid spacing.

The question arises whether a more quantitative terrain classification is possible for the present purpose. An attempt is made using the "normalized" variance of the terrain. The variances of the terrain heights (see also table 3) are obtained from the residuals left after fitting a 2nd order polynomial through the observed heights in a network of 20 x 20 points. This "trend" has been subtracted from the raw data to obtain a zero mean and to remove global landforms which are insignificant for the result of interpolation. Of interest are only local variations.

The polynomial is computed from a given number of points. Therefore the area, over which it is found, is different for the small and large scale terrain models. In the case of small scale models, the polynomial is determined for an area of $800 \times 800\ m^2$. In large scale models this is only $200 \times 200\ m^2$. Consequently, an identical terrain model would produce different polynomials and variances depending on whether they are found from 200 x 200 or from 800 x 800 m^2.

For use as an indicator for the coefficient a_1 of relation (15), the effect of the size of the area has to be removed from the trend polynomial from which it was determined. This is attempted by "normalising" the standard deviations σ_t, by simply dividing these by the side length s of the area considered. Thus one obtains normalized standard deviations σ_n:

$$\sigma_n = \sigma_t / s \qquad \text{(16)}$$

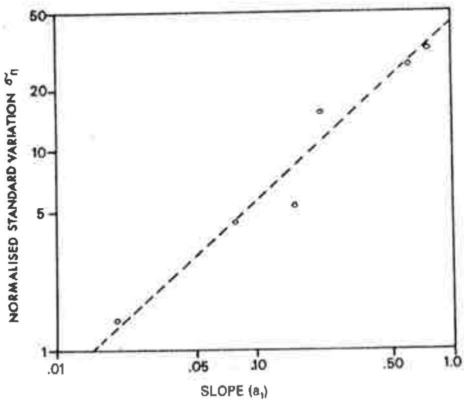

Fig. 10. Normalized standard variation of terrain versus slope a_1 of the linea regression formula (15).

Figure 10 shows that there is high correlation between the normalized σ_n and the coefficient a_1 of relation (15) for the six terrain models under consideration.

Provided that further experience confirms the present results, relation (15), together with figure 10, could be used as an accuracy model in planning for a DTM, according to the following method:

- An estimate is made of the local variation of the relief of the terrain in question, i.e. with the standard deviation σ_t of residuals after polynomial smoothing. The size of the area is described by side length s.
- The normalized standard deviation $\sigma_n = \sigma_t/s$ is computed. From figure 10, an estimate of coefficient a_1 is obtained.
- Coefficient a_0 is found as the standard measuring error.
- Evaluation of relation (15) for a given grid spacing, provides an estimate of the accuracy of terrain representation. If this accuracy is specified, then the grid spacing to obtain such accuracy can be computed.

This procedure is of course only applicable for those ranges of σ_n and grid spacing, for which figure 10 and relation (15) have been established. Plotting the variance of the terrain versus interpolation accuracy leads to figure 11 and a slightly different approach to the prediction of accuracy of terrain representation.

Both axes of figures 11 are in logarithmic scales.

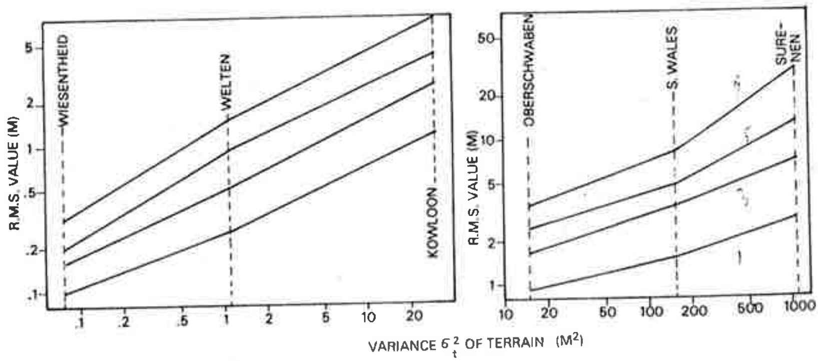

Figure 11. Relation between terrain variance and DTM performance. (The scales are logarithmic)

With these scales, the relation between variances and e-values are again approximately linear. Almost exact linearity is obtained in the large scale cases (i.e. fig. 11a, where variances vary between 0.08 m^2 up to 30 m^2). In the three small scale models, a larger range of variances is covered, from 15 m^2 up to 1,021 m^2. Over this larger range, slight non-linearity of the relation (with logarithmic scales) is found.

This suggests the following regression function between variance σ_t^2 of terrain height (according to above definition) and the r. m. s. interpolation error:

$$e = 10^{b_o + b_1 \log \sigma_t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

Coefficient b_o is clearly a function of the grid spacing, whereas coefficient b_1 appears to be almost constant in the cases under study.

The accuracy models according to formulae (15), (16), and figure 10, do represent to some extent an alternative to the approach of MAKAROVIČ [23] , in so far as they both could serve as a tool for planning of a DTM. In [23] accuracy models were produced analytically, starting from the smallest topographic feature, which should still be reconstructed from the sampled terrain points rather than from a terrain type and desired accuracy. Therefore, this approach is rather different from the experimental one presented here. It is left for further study to establish the relation between the two approaches.

Summarizing, the conclusions on the relation between terrain types and accuracy are:

- The normalized standard deviation of terrain height appears to be an indicator for the relation "grid spacing - DTM accuracy".
- The slope of the linear regression (15) (between DTM accuracy and grid spacing) is strongly correlated with the normalized standard deviation of terrain relief.
- With logarithmic scales, the regression between DTM accuracy and terrain variance is approximately linear.
- The slope of this relation (accuracy/terrain variance) is constant, irrespective of grid spacing. Grid spacing only displaces the regression line, but does not affect its inclination.

4.6 Variation of Accuracy as a Function of Planimetric Location

This problem consists of two components. Firstly the question arises as to how the interpolation accuracy varies within a square mesh, using the identical reference points. The second question concerns the variation of accuracy over larger distances within a whole DTM.

Grid spacing (m)	30		50		90		135		225		405	
Point	P2	P3	P2	P3	P2	P3	P2	P3	P2	P3	P2	P3
Wiesentheid	.94	.82	.84	.56	.60	.29						
Oberschw.							.84	.61	.73	.43	.76	.34
Welten	.98	.61	.62	.25	1.17	.31						
S. Wales							.99	.56	.87	.34	.80	.16
Kowloon	.86	.62	.63	.31	.88	.26						
Surenen							.80	.54	.73	.33	.72	.19
Average	.94	.68	.69	.37	.88	.29	.88	.57	.78	.37	.76	.23

Table 18.

Variation of interpolation error within a square mesh of varying side length. Shown are e_2/e_1, e_3/e_1 -values, relative to the centre of gravity. P2. P3 are the eccentric points; according to figure 5. Point P3 is at a constant distance from the closest reference point, whereas for points P1, P2, the distance to the closest points increases with increasing grid spacing

An answer to the first question may be found from table 18. This shows the e_2, e_3-values found with method LP, for the two eccentric points (see fig. 5). These values are normalized using the e_1-value of the center of gravity.

The accuracy of interpolation obviously improves considerably if a point in the vicinity of a reference point is interpolated, rather than the center of gravity of a symmetric pattern. If the eccentric point is kept at a constant distance from the closest reference point then this effect is pronounced with increasing grid spacing. Therefore, the accuracy of the interpolation of an eccentric point close to a reference point hardly depends on the grid spacing.

The second problem, of varying accuracy within the whole of the DTM, is illustrated in figure 12. For the center of gravity and the grid spacings indicated in the figure the example shows the interpolation errors of 16 x 16 points of the six models.

The magnitude of the errors v is described by symbols "1", "2", "3", and "4":

$$v \leq e/2 \rightarrow 1$$
$$e/2 \leq v \leq e \rightarrow 2$$
$$e \leq v \leq 2e \rightarrow 3$$
$$2.e < v \rightarrow 4$$

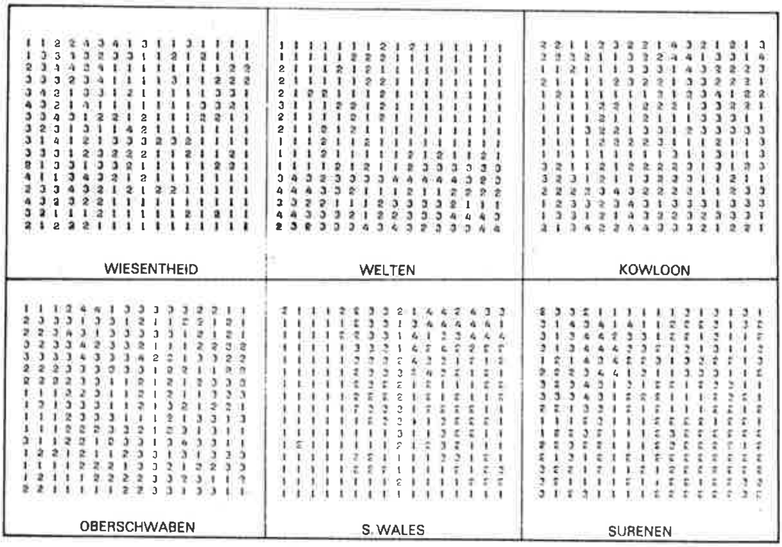

Figure 12. Error distribution at the centres of gravity in the 6 terrain models for a grid spacing $g = 3$.

The 6 selected graphs show a fairly even distribution of the various error magnitudes in most of the models. Strong clustering of errors of similar magnitude occur only in models "Welten" and "South Wales". Visual inspection of the error distributions for varying grid spacing and planimetric location allows the following conclusions to be drawn:

- The distance to the closest reference point does not affect the distribution of error magnitudes. The error magnitudes are either clustered for all three investigated planimetric locations, or for none of them..
- The grid spacing has no significant influence on the clustering of error magnitudes since the error magnitudes are either clustered for all investigated grid spacings of a terrain model or for none of them.
- No tendency is noticed for an increase (or decrease) of clustering with increase of the variance σ_t^2 of the terrain.
- Except for models "Welten" and "South Wales" the area covered by the individual terrain samples of the experiment appears to be sufficiently small since no excessive clustering occurs of errors of the same class. (No large variation of terrain type occurs within the sample).

Terrain models "Welten" and "South Wales" demonstrate inhomogeneity of accuracy. In figure 12, the upper section of model "Welten" and the lower left part of model "South Wales" should have been sampled with a different grid spacing than the plot of the area. This emphasises the fact that problems of adjusting sampling density do represent an important field for further study.

5. CONCLUSIONS AND RECOMMENDATIONS

5.1. Conclusions

The present study was limited to the problem of interpolation in square grid DTMs. By means of a numerical experiment with 6 different terrain samples, the study concentrated on the performance of various interpolation algorithms, and the relation between accuracy of terrain representation, sampling density, and type of terrain.

Interpolation methods were studied in three respects: weighting, accuracy, and computation time. Linear prediction and patchwise polynomial interpolation were found to be the most effective interpolation methods for the specific case of square grid DTMs. The method of moving averages is of the same accuracy as the two aforementioned algorithms, but somewhat more expensive. The average improvement obtained by the use in the experiment of one of these methods rather than of simple linear interpolation was found to be in the order of magnitude of 20 - 30%, with a maximum of approximately 50%.

Weighting deserves utmost attention when applying moving averages, or linear prediction. Incorrect weight might even have a detrimental effect on the performance of these methods. As far as linear prediction is concerned, it was concluded that computation of a second order polynomial "trend" was a necessity to obtain satisfactory results of interpolation.

For certain applications, the use of a patchwise polynomial surface rather than the pointwise interpolation algorithms might be a significant advantage (profiling, contouring, intersection with lines). It has been found that for such cases, the bilinear polynomial through 4 reference points, and the more sophisticated patchwise third degree polynomial do represent valuable alternatives to linear prediction, with almost the same performance.

It was further concluded, that no gain is to be expected by using more than the 4 x 4 surrounding reference points for interpolation of a new point. It also became obvious, however, that the use of 4 x 4 points is slightly superior (10%) to the use of only the 4 closest reference points.

It could finally be shown, that a linear relation exists between interpolation accuracy and sampling density. The slope of this linear regression is related to the terrain type. In an attempt to identify an indicator for the slope of the regression, it was found that it is correlated with the "normalized standard deviation of terrain relief", as defined in formula (16). Based on these results, an accuracy model is tentatively proposed to predict interpolation errors in future projects.

5.2. Recommendations

Traditionally one uses linear interpolation in regular grid DTMs [3] , [35], [33/3] . The present study leads to the recommendation not to oversimplify interpolation in square grid DTMs. Instead, linear prediction or patchwise polynomials should be considered, depending on the use of the DTM and the number of points to be interpolated per mesh.

For objective decisions on developments of future DTMs, information is of course required beyond the results of the present study. The most important questions to be answered apart from those on square grid DTMs concern:
adjustment of sampling density; classification of terrain relief; elimination of gross errors; inclusion of terrain break lines and points in the DTM, which is procured along contours or on a regular grid; continuous versus point by point sampling of lines; and sampling along contours versus square grid DTM.

The study of these problems presumably leads to rather extensive numerical work, similar to the one presented in this report. It is highly desirable to keep a common frame for objective comparison of alternatives. The organisation of future studies should have as an important goal the comparability of their results with previous ones.

ACKNOWLEDGEMENT

The author wishes to thank Mr. W. JIWALAI for supervising the acquisition of measurements, and for his help in clarifying a number of problems during the initial stage of the study.

REFERENCES

[1] ANON — "Digital Terrain Model - A Presentation of the Practical Application of Some Existing DTM", ISP, Intercomm. Working Group IV/V, Int. Conference at Lausanne, 1968.

[2] ANON — "Photogrammetry in Highway Design", ISP, Working Group Report IV/1, Int. Conference at Ottawa, 1972.

[3] ANON — "GPCP - A General Purpose Contouring Program", California Computer Products, Inc., 1968.

[4] ARTHUR, D.W.G. — "Interpolation of a Function of Many Variables", Photogramm. Engineering, 1965/2.

[5] ARTHUR, D.W.G. — "Interpolation of a Function of Many Variables II", Photogrammetric Engineering, 1973/2.

[6] BEYER, A. — "Zur Erfassung flachen Geländes durch willkürlich verteilte Höhenpunkte, Vermessungstechnik, 1972/7.

[7] BENNER, E., SCHULT, L. — "Report on a Digital Terrain Model", Pres. Paper, ISP-Conference, Ottawa, 1972.

[8] BLASCHKE, W. — "Problems in Obtaining and Processing of Photogrammetric Data for DTMs", Inv. Paper, ISP Conference, Lausanne, 1968.

[9] BOSMAN, E.R. ECKHART, D., KUBIK, K. — "Delft - A Programme System for Surface Approximation" Data Processing Department, The Hague, The Netherlands, 1971.

[10] GOTTHARDT, E. — "Eine Interpolationsformel für die Modell einpassung", Bildmessung und Luftbildwesen, 1962/2.

[11] HARDY, R. — "Multiquadric Equations of Topography and Other Irregular Surfaces", J. Geophysical Res., 76/8, 1971.

[12] HARDY, R. — "Geodetic Application of Multiquadric Analysis", Allgemeine Vermessungsnachrichten, 1972/10.

[13] JAINCAITIS, J.E. JUNKINS, J.L. — "Modeling Irregular Surfaces", Photogrammetric Engineering, 1973/4.

[14] JIWALAI, W. — "Digital Terrain Models - numerical investigation on regular point grids". Unpublished MSc Thesis, ITC, 1972.

[15] KEIR, K.M. — "Photogrammetry and Current Practice in Road Design", ISP Conference, Working Group IV/1, Ottawa, 1972.

[16] KRAUS, K., MIKHAIL, E.M. — "Linear Last Squares Interpolation", Photogrammetric Engineering, 1972/10.

[17] KRAUS, K. — "Automatische Berechnung digitaler Höhenlinien", Bildmessung und Luftbildwesen, 1972.

[18] LAUER, S. — "Anwendung der skalaren Prädiktionen auf das Problem des Digitalen Geländemodelles", Nachrichten aus dem Karten- und Vermessungswesen, Frankfurt, 1972

[19] LEBERL, F. — "Einfache und verknüpfte Polynome zur Darstellung von Verzeichnungskurven" Bildmessung und Luftbildwesen, 1970/2.

[20] LINKWITZ, K. — "Digital Terrain Models", Bildmessung und Luftbildwesen, 1970/1.

[21] LINKWITZ, K., RUMMEL, J. — "Ein kontinuierbildes digitales Geländemodell dargestellt am Beispiel des Deutschen Pavillons Montreal", Bildmessung und Luftbildwesen, 35, 1967.

[22] MAKAROVIČ, B. — "The Applications of Automated Equipment in Photogrammetry", ITC Publ. A53, 1972.

[23] MAKAROVIČ, B. — "Reconstruction of Data from Sampled Points", Photogrammetria, Oct. 1972.

[24] MAKAROVIČ, B. — "Progressive Sampling for Digital Terrain Models", ITC Journal, 1973/3.

[25] NAKAMURA, H. — "On Digital Terrain Models", ISP-Conference, Lausanne, 1968.

[26] SCHATZ, U. — "Das Problem der optimalen Stützpunktdichte und der optimalen Maschengrösse bei Transformation ungleichartiger Koordinaten", Ph. D. Thesis, Univ. Bonn, W.Germany, 1970.

[27] SCHUT, G. H. "External Block Adjustment of Planimetry", Photogrammetric Engineering, 1970/9.

[28] SCHWEBEL, R. "Measurement of Digital Terrain Model", XIIIth Fig.- Congress, Wiesbaden, W. Germany, 1971.

[29] SILAR, F. "Das digitale Geländemodell - Theorie und Praxis", Vermessungtechnik, 1972/9.

[30] SILAR, F. "Die Interpolation der Höhen auf dem DGM in Beziehung fur Klassifikation der Geländeflachen und zur Dichte und Zerlegung der Knotenpunkte ISP-Working Group IV/1, Bratislava, 1969.

[31] SIMA, J. "Principien des čs digitalen Geländemodells", Vermessungstechnik, 1972/2

[32] SIMA, J. "Analysis of some Existing DTM Systems", ISP-Working Group IV/1, Bratislava, 1969.

[33] SOCIETÉ FRANCAISE DE PHOTOGRAMMETRIE "Photogrammetrie et Etudes d'Autoroutes", Bulletin No. 17, 1967. with the following contributions:

[33/1] BLASCHKE, W. "Le Modèle Digitale M.I.T.",

[33/2] NORDIN, H. "La Représentation digitale du terrain",

[33/3] VIITA, E. "La mise en point, en Finlande, de la méthode par surfaces",

[33/4] HALLMEN, B. "La représentation Nordisk ADB du terrain",

[33/5] DELIGNY, J. L. "La méthode du semis de points".

[34] STASTNY, V. "On the Development of the System of Programming the Formation and Exploitation of the DTM", ISP-Working Group IV/1, Bratislava, 1969.

[35] STEWARDSON, P. B., GSELL, D. C., KRAUS, K. "DACS - Digital Automatic Contouring System", Pres. Paper, ISP-Conference, Ottawa, 1972.

[36] TERNRYD, C. O. "Development of the Application of Photogrammetry to Highway Design during 1969 - 1968", Inv. Paper, ISP-Conference, Working Group IV/1, Lausanne 1968.

[37] VLCEK, J. "Adjustment of Strips Using Orthogonal Polynomials", Photogrammetric Engineering, Vol. 31/2, 1965.

[38] WOLF, H. "Ausgleichsrechnung nach der Methode der kleinsten Quadrate", Dümmler Verlag, Bonn, 1968.

[39] YAGLOM, A. M. "Introduction to the Theory of Stationary Random Functions", Prentice Hall, Inc., 1962.

[40] KRAUS, K. "Digitale Höhenlinienkartierung", Photogram. Week, Stuttgart, 1973.

[41] CRANE, I. K. "Computer interpolation and contouring from two-dimensional data", Geoexploration, Vol. 8 (1970).

[42] AHLBERG, J. H., NILSON, E. N., WALSH, J. L. "The theory of splines and their applications", Academic Press, New York, 1967.