Computational Mechanics 20 (1997) 452–459 © Springer-Verlag 1997

A new visco- and elastodynamic time domain Boundary Element formulation

M. Schanz, H. Antes

Abstract The usual time domain Boundary Element Method (BEM) contains fundamental solutions which are convoluted with time-dependent boundary data and integrated over the boundary surface. If the fundamental solution is known, e.g., in Elastodynamics, the temporal convolution can be performed analytically when the boundary data are approximated by polynomial shape functions in time and in the boundary elements. This formulation is well known, but the resulting time-stepping BEM procedure produces instabilities and high numerical damping, when the time step size is chosen too small and too large, respectively. Moreover, in case of viscoelastic or poroelastic domains, the fundamental solution is known only in the frequency domain such that the time history of a response can only be obtained by an inverse transformation of the frequency domain results.

Here, a new approach for the evaluation of the convolution integrals, the so-called "Operational Quadrature Methods" developed by Lubich, is presented. In this formulation, the convolution integral is numerically approximated by a quadrature formula whose weights are determined by the Laplace transform of the fundamental solution and a linear multistep method. Hence, the frequency domain fundamental solution can be used without the need of an inverse transformation. Therefore, the extension to viscoelastic problems succeeds using the elastic-viscoelastic correspondence principle.

1 Introduction

The Boundary Element Method (BEM) has become a widely used numerical method. A review about the efforts in elastodynamics with BEM is published, e.g., by Beskos (1987). The main advantage of the method is the reduction of the problem dimension by one. This means, for a three dimensional problem only the surface of the domain must be discretized, opposite to domain methods like the Finite Element or Finite Difference Method.

In case of transient elastodynamic problems, the BEM is mostly used in frequency or Laplace domain followed by an inverse transformation, e.g., Ahmad and Manolis (1987). Mansur (1983) developed one of the first Boundary Element formulations directly in the time domain for the scalar wave equation and for elastodynamics with zero initial conditions (Mansur and Brebbia 1983). The extension of this formulation to non-zero initial conditions was presented by Antes (1985). Detailed information about this procedure may be found in the book of Domínguez (1993). As possible applications, one should mention soil-structure-interaction problems (e.g., Banerjee, Ahmad and Wang 1989), dynamic analysis of 3-D foundation (Karabalis and Rizos 1993), contact problems (Antes and Panagiotopoulos 1992) and viscoelastic problems (Gaul and Schanz 1994b).

All formulations in time domain, however, require an adequate choice of the time step size. An improper chosen time step size leads to instabilities or numerical damping. A first improvement of this behaviour was proposed by Antes and Jäger (1995) for acoustics and by Schanz, Gaul and Antes (1993) for elastodynamics. Further improvements are presented in the papers of Rizos and Karabalis (1994) and of Coda and Venturini (1995). Another disadvantage of the formulation in time domain is, however, that not for all physical problems time-dependent fundamental solutions are known in an explicit analytical form, e.g., in poroelasto dynamics (Wiebe and Antes 1991).

Therefore, here, a BE formulation in time domain is presented which is based on the "Operational Quadrature Methods" published by Lubich (1988a). This method is a quadrature formula which approximates the convolution integral in the time domain boundary integral equation. The quadrature weights are determined from the fundamental solutions of the Laplace domain. In Sect. 2, this quadrature method is summarised. In the next Sect. 3, the boundary element formulations for elastodynamics and viscoelasticity are developed using this quadrature method. Numerical results of a 3-D bar are presented to show the accuracy of the procedure.

2 Convolution quadrature

In the following, the convolution in the time domain boundary integral equation is approximated by the so-called "Operational Quadrature Method" developed by Lubich (1988a). As a first step, in the convolution integral

Communicated by D.E. Beskos, 2 March 1997

M. Schanz, H. Antes
Institute of Applied Mechanics,
Technical University of Braunschweig, Spielmannstr. 11,
D-38106 Braunschweig, Germany

Correspondence to: M. Schanz

Dedicated to Professor Dr.-Ing. F. Ziegler on the occasion of his 60th birthday

$$y(t) = f(t) * g(t) = \int_0^t f(t-\tau)\ g(\tau)\ \mathrm{d}\tau \tag{1}$$

the inverse Laplace transform

$$\mathscr{L}^{-1}\{\hat{f}(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(s)e^{st}\mathrm{d}s = \begin{cases} f(t) & t > 0 \\ 0 & t < 0 \end{cases} , \tag{2}$$

with

$$\hat{f}(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t \tag{3}$$

is used to substitue $f(t-\tau)$. Exchanging the integration sequence yields finally

$$\int_0^t f(t-\tau)\ g(\tau)\ \mathrm{d}\tau = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(s)\underbrace{\int_0^t e^{s(t-\tau)}g(\tau)\,\mathrm{d}\tau}_{x(t)}\ \mathrm{d}s\ . \tag{4}$$

Then, as a second step, the inner integral, abbreviated with $x(t)$, will be approximated. Since $x(t)$ is a solution of the first order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = sx(t) + g(t) \text{ with } x(0) = 0\ , \tag{5}$$

$x(t) = x(n\Delta t)$ can be found approximatively by determining x_n from a linear multistep method

$$\sum_{j=0}^{k} \alpha_j x_{n-j} = \Delta t \sum_{j=0}^{k} \beta_j (s x_{n-j} + g((n-j)\ \Delta t)) \tag{6}$$

with equal time steps Δt, and the starting values $x_{-k} = \ldots = x_{-1} = 0$. In this representation of the multistep method it is not possible to extract the discrete value x_n to be inserted in Eq. (4). Therefore, Eq. (6) is multiplied with $z^n (z \in \mathbb{C})$ and summed up over n from 0 to ∞. This leads to

$$\begin{aligned}\sum_{n=0}^{\infty}[\alpha_0 x_n + \alpha_1 x_{n-1} + \ldots + \alpha_k x_{n-k}]z^n \\ = \Delta t \sum_{n=0}^{\infty}[\beta_0(s x_n + g(n\Delta t)) + \cdots \\ + \beta_k(s x_{n-k} + g((n-k)\Delta t))]z^n\end{aligned} \tag{7}$$

and finally, assuming $g(t<0) = 0$, with (formal) power series for $x(t) = \sum_{n=0}^{\infty} x_n z^n$ and $g(t) = \sum_{n=0}^{\infty} g(n\Delta t)z^n$ to

$$\begin{aligned}[\alpha_0 + \alpha_1 z + \ldots + \alpha_k z^k]\sum_{n=0}^{\infty} x_n z^n \\ = \Delta t[\beta_0 + \ldots + \beta_k z_k] \\ \times \left[s\sum_{n=0}^{\infty} x_n z^n + \sum_{n=0}^{\infty} g(n\Delta t) z^n\right]\end{aligned} \tag{8}$$

or, with

$$\gamma(z) = \frac{\alpha_0 + \ldots + \alpha_k z^k}{\beta_0 + \ldots + \beta_k z^k} \tag{9}$$

to

$$\left(\frac{\gamma(z)}{\Delta t} - s\right)\sum_{n=0}^{\infty} x_n z^n = \sum_{n=0}^{\infty} g(n\Delta t)z^n\ . \tag{10}$$

Obviously, the applied multistep method is characterised by $\gamma(z)$. Hence, the conditions of being $A(\alpha)$-stable with positive angle α, stable in a neighbourhood of infinity, strongly zero-stable and consistent of order p, (see Lubich and Schneider 1992) can be stated as follows:

- $\gamma(z)$ has neither zeros nor poles on the closed unit disk ($|z| \leq 1$), with the exception of a simple zero at $z = 1$, (11)
- $|\arg\gamma(z)| \leq \pi - \alpha$, with $|z| \leq 1$, for $\alpha > 0$, (12)
- $\Delta t^{-1}\gamma(e^{-\Delta t}) =\ 1 + O(\Delta t^p)$, with $\Delta t \to 0$, for $p \geq 1$. (13)

Well known examples of possible $\gamma(z)$ are the backward differentiation formulas of order $p \leq 6$, e.g., of order 2 given by $\gamma(z) = \frac{3}{2} - 2z + \frac{1}{2}z^2$.

With the approximation (10) of $x(t)$, the convolution integral (1) is approximated by

$$\sum_{n=0}^{\infty} y(n\Delta t)z^n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{f}(s)\frac{1}{\frac{\gamma(z)}{\Delta t} - s}\mathrm{d}s \times \sum_{n=0}^{\infty} g(n\Delta t)z^n\ . \tag{14}$$

The integration along the curve $c - i\infty$ to $c + i\infty$ is changed to a closed contour by adding a half circle at its ends (Fig. 1). If the function $\hat{f}(s)$ satisfies the assumption

$$|\hat{f}(s)| \to 0 \text{ for } \Re(s) \geq c \text{ and } |s| \to \infty\ , \tag{15}$$

the integral in (14) can be determined by CAUCHY's integral formula, i.e.,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{f}(s)\frac{1}{\frac{\gamma(z)}{\Delta t} - s}\mathrm{d}s\sum_{n=0}^{\infty} g(n\Delta t)z^n = \hat{f}\left(\frac{\gamma(z)}{\Delta t}\right)\sum_{n=0}^{\infty} g(n\Delta t)z^n\ . \tag{16}$$

Then, by expressing the function $\hat{f}(z)$ as a power series

$$\hat{f}\left(\frac{\gamma(z)}{\Delta t}\right) = \sum_{n=0}^{\infty}\omega_n(\Delta t)z^n\ , \tag{17}$$

with the coefficients

$$\omega_n(\Delta t) = \frac{1}{2\pi i}\int_{|z|=\tau}\hat{f}\left(\frac{\gamma(z)}{\Delta t}\right)z^{-n-1}\mathrm{d}z\ , \tag{18}$$

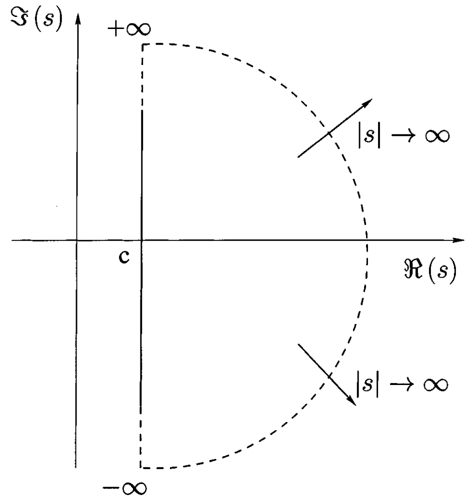

Fig. 1. Integration path of integral in Eq. (14)

and τ being the radius of a circle in the domain of analyticity of $\hat{f}(z)$, Eq. (16) can be modified by CAUCHY's product of two series (Bronstein and Semendjajew 1984) as

$$\sum_{n=0}^{\infty} \omega_n(\Delta t) z^n \sum_{n=0}^{\infty} g(n\Delta t) z^n = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \omega_{n-j}(\Delta t) g(j\Delta t) z^n \quad . \tag{19}$$

Since the multiplication with z^n and the subsequent summation over n in (14) was introduced only for enabling the extraction of x_n in Eq. (10), the final formula needs only the n-th coefficient of (19). This leads to the following quadrature expression:

$$y(n\Delta t) = \sum_{j=0}^{n} \omega_{n-j}(\Delta t) g(j\Delta t), \; n = 0, 1, \ldots, N \quad . \tag{20}$$

The integration weights ω_n can be determined by evaluating Eq. (18). After transforming to polar coordinates, this integral is approximated by a trapezoidal rule with L equal steps $\frac{2\pi}{L}$ (Lubich 1988b)

$$\omega_n(\Delta t) = \frac{\tau^{-n}}{L} \sum_{\ell=0}^{L-1} \hat{f}\left(\frac{\gamma\left(\tau e^{i\ell\frac{2\pi}{L}}\right)}{\Delta t} \right) e^{-in\ell\frac{2\pi}{L}}, \quad n = 0, 1, \ldots, N \quad . \tag{21}$$

From (21), using the technique of the Fast Fourier Transformation (FFT), the weights ω_n can be calculated very fast. If one assumes that the values of $\hat{f}(z)$ in (21) are computed with an error bound ϵ, the choice $L = N$ and $\tau^N = \sqrt{\epsilon}$ yields an error in ω_n of order $O(\sqrt{\epsilon})$, (see Lubich 1988b).

3 BE formulation

3.1 Elastodynamic formulation

Assuming homogeneity and linearly elastic material (Young's modulus E, mass density ϱ and Poisson's ratio ν), the dynamic behaviour of a structure as well as the wave propagation in a domain Ω is governed by the equation of motion

$$\left(c_1^2 - c_2^2\right) u_{i,ij} + c_2^2 u_{j,ii} + \frac{b_j}{\varrho} = \ddot{u}_j \tag{22}$$

where $u_j = u_j(\mathbf{x}, t)$ is the displacement vector at point $\mathbf{x}$ and time t, b_j is the body force vector per unit mass, c_1 and c_2 are the dilatational and shear wave velocities, respectively:

$$c_1^2 = \frac{E(1-\nu)}{\varrho(1-2\nu)(1+\nu)}, \quad c_2^2 = \frac{E}{\varrho(1+\nu)^2} \quad . \tag{23}$$

In the above equations, Latin indices receive the values 1,2, and 1,2,3 in 2-D and 3-D, respectively, where summation convention is implied over repeated indices, and commas and over-dots denote spatial and temporal differentiation, respectively. On the boundary $\Gamma = \Gamma_u \cup \Gamma_p$ of the domain Ω, tractions p_i and displacements u_i, respectively, i.e.

$$\begin{aligned} \sigma_{ij} n_j &= p_i(\mathbf{x}, t) \quad \text{for } t > 0, \mathbf{x} \in \Gamma_p, \\ u_i(\mathbf{x}, t) &= q_i(\mathbf{x}, t) \text{ for } t > 0, \mathbf{x} \in \Gamma_u \end{aligned} \tag{24}$$

are prescribed, while the initial conditions $u_i(\mathbf{x}, 0)$ and $\dot{u}_i(\mathbf{x}, 0), \mathbf{x} \in \Omega \cup \Gamma$ are assumed to be zero

$$u_i(\mathbf{x}, 0) = 0, \quad \dot{u}_i(\mathbf{x}, 0) = 0 \quad \mathbf{x} \in \Omega \cup \Gamma \quad . \tag{25}$$

σ_{ij} is the stress tensor and n_j means the outward normal vector on the boundary Γ. Neglecting the body force effects, the dynamic extension of Betti's reciprocal work theorem combining two states of displacements and tractions (U_{ij}, T_{ij}) and (u_j, p_j) leads to the integral equation (Eringen and Suhubi 1975)

$$c_{ij}(\mathbf{y}) u_j(\mathbf{y}, t) = \int_{\Gamma_x} U_{ij}(\mathbf{x}, \mathbf{y}, t) * p_j(\mathbf{x}, t) \, d\Gamma_x - \oint_{\Gamma_x} T_{ij}(\mathbf{x}, \mathbf{y}, t) * u_j(\mathbf{x}, t) \, d\Gamma_x \tag{26}$$

where $*$ denotes the convolution with respect to time. U_{ij} and T_{ij} are the displacements and tractions, respectively, due to a unit impulse in the direction x_i, i.e., the time-dependent fundamental solution of the full space. The integral free terms $c_{ij}(\mathbf{y})$ are identical to those of elastostatics and dependent only on the local geometry at $\mathbf{y}$ and on Poisson's ratio ν, for $\mathbf{y}$ on a smooth boundary $c_{ij}(\mathbf{y}) = \frac{\delta_{ij}}{2}$. When $\mathbf{x}$ approaches $\mathbf{y}$, the kernel $U_{ij}(\mathbf{x}, \mathbf{y}, t)$ is weakly singular and $T_{ij}(\mathbf{x}, \mathbf{y}, t)$ is strongly singular, i.e., the second integral in Eq. (26) exists only in the Cauchy principal value sense.

According to the boundary element method the boundary surface Γ is discretized by E iso-parametric elements Γ_e where F polynomial spatial shape functions $N_e^f(\mathbf{x})$ are defined. Hence, with the time dependent nodal values $u_j^{ef}(t)$ and $p_j^{ef}(t)$ the following representation is adapted

$$\begin{aligned} u_j(\mathbf{x}, t) &= \sum_{e=1}^{E} \sum_{f=1}^{F} N_e^f(\mathbf{x}) u_j^{ef}(t), \\ p_j(\mathbf{x}, t) &= \sum_{e=1}^{E} \sum_{f=1}^{F} N_e^f(\mathbf{x}) p_j^{ef}(t) \quad . \end{aligned} \tag{27}$$

Inserting these 'ansatz' functions in Eq. (26) gives

$$c_{ij}(\mathbf{y})u_j(\mathbf{y},t) = \sum_{e=1}^{E}\sum_{f=1}^{F}\left\{\int_{\Gamma_e} U_{ij}(\mathbf{x},\mathbf{y},t)N_e^f(\mathbf{x})\ \mathrm{d}\Gamma_x * p_j^{ef}(t) - \oint_{\Gamma_e} T_{ij}(\mathbf{x},\mathbf{y},t)N_e^f(\mathbf{x})\ \mathrm{d}\Gamma_x * u_j^{ef}(t)\right\} \ . \tag{28}$$

When the time-period t is discretized by N equal time-increments Δt, the convolution-integral between the fundamental solutions U_{ij} or T_{ij} and the nodal values $p_j^{ef}(t)$ or $u_j^{ef}(t)$, respectively, may be approximated by the Convolution Quadrature formula (20). The result is the following new boundary element time-stepping formulation for $n = 0, 1, \ldots, N$

$$c_{ij}(\mathbf{y})u_j(\mathbf{y},n\Delta t) = \sum_{e=1}^{E}\sum_{f=1}^{F}\sum_{k=0}^{n}\left\{\omega_{n-k}^{ef}(\hat{U}_{ij},\mathbf{y},\Delta t)p_j^{ef}(k\Delta t) - \omega_{n-k}^{ef}(\hat{T}_{ij},\mathbf{y},\Delta t)u_j^{ef}(k\Delta t)\right\} \tag{29}$$

with the weights corresponding to Eq. (21)

$$\omega_n^{ef}(\hat{U}_{ij},\mathbf{y},\Delta t) = \frac{\tau^{-n}}{L}\sum_{\ell=0}^{L-1}\int_{\Gamma_e}\hat{U}_{ij}\left(\mathbf{x},\mathbf{y},\frac{\gamma\left(\tau e^{i\ell\frac{2\pi}{L}}\right)}{\Delta t}\right) \times N_e^f(\mathbf{x})\ \mathrm{d}\Gamma_x\ e^{-in\ell\frac{2\pi}{L}} \ , \tag{30}$$

$$\omega_n^{ef}(\hat{T}_{ij},\mathbf{y},\Delta t) = \frac{\tau^{-n}}{L}\sum_{\ell=0}^{L-1}\oint_{\Gamma_e}\hat{T}_{ij}\left(\mathbf{x},\mathbf{y},\frac{\gamma\left(\tau e^{i\ell\frac{2\pi}{L}}\right)}{\Delta t}\right) \times N_e^f(\mathbf{x})\ \mathrm{d}\Gamma_x\ e^{-in\ell\frac{2\pi}{L}} \ , \tag{31}$$

Note that the calculation of the quadrature weights (30) and (31) is only based on the Laplace transformed fundamental solutions. Therefore, in elastodynamics

$$\hat{U}_{ij}(\mathbf{x},\mathbf{y},s) = \frac{1}{4\pi\varrho}\left\{\frac{1}{r^2}\left(\frac{3r_ir_j}{r^3}-\frac{\delta_{ij}}{r}\right) \times\left[\frac{s\frac{r}{c_1}+1}{s^2}e^{-\frac{r}{c_1}s}-\frac{s\frac{r}{c_2}+1}{s^2}e^{-\frac{r}{c_2}s}\right] + \frac{r_ir_j}{r^3}\left[\frac{1}{c_1^2}e^{-\frac{r}{c_1}s}-\frac{1}{c_2^2}e^{-\frac{r}{c_2}s}\right]+\frac{\delta_{ij}}{rc_2^2}e^{-\frac{r}{c_2}s}\right\} \tag{32}$$

can be used (Cruse and Rizzo 1968) in (30), where $r = \sqrt{r_ir_i}$ with $r_i = x_i - y_i$. The corresponding fundamental traction components, needed in Eq. (31), are obtained from Eq. (32) via

$$\hat{T}_{ij} = \varrho\left(c_1^2 - 2c_2^2\right)\hat{U}_{jm,m}\delta_{ik}n_k + \varrho c_2^2\left(\hat{U}_{ji,k}n_k + \hat{U}_{jk,i}n_k\right) \tag{33}$$

with the Kronecker symbol δ_{ik}. In Eqs. (30) and (31) the integrations over each boundary element Γ_e are usually performed by Gaussian quadrature; only when $\mathbf{x}$ approaches $\mathbf{y}$ the respective integral of (31) is evaluated following the procedure proposed by Guiggiani and Gigante (1990). In order to arrive in Eq. (29) at systems of algebraic equations, collocation is used at every node of the shape functions $N_e^f(\mathbf{x})$, and, finally, a direct equation solver is applied.

3.2
Viscoelastic formulation

The above presented procedure can also be achieved in viscoelasticity by simply exchanging the fundamental solution in (30) and (31) by the corresponding viscoelastic ones.

3.2.1
Viscoelastic constitutive equations

When decomposing the stress tensor σ_{ij} into the hydrostatic part $\delta_{ij}\sigma_{kk}/3$ and the deviatoric part s_{ij}

$$\sigma_{ij} = \tfrac{1}{3}\sigma_{kk}\delta_{ij} + s_{ij} \quad \text{with } s_{ii} = 0 \ . \tag{34}$$

and correspondingly the strain tensor ε_{ij} in

$$\varepsilon_{ij} = \tfrac{1}{3}\varepsilon_{kk}\delta_{ij} + e_{ij} \quad \text{with } e_{ii} = 0 \ , \tag{35}$$

two independent sets of constitutive equations exist for viscoelastic materials

$$\sum_{k=0}^{M} a_k\frac{d^k}{dt^k}s_{ij} = \sum_{k=0}^{M} b_k\frac{d^k}{dt^k}e_{ij} \text{ and } \sum_{k=0}^{M} c_k\frac{d^k}{dt^k}\sigma_{ii} = \sum_{k=0}^{M} d_k\frac{d^k}{dt^k}\varepsilon_{ii} \ , \tag{36}$$

where a_k, b_k, c_k and d_k are the parameters of the viscoelastic model. More flexibility in fitting measured data in a large frequency range is obtained by replacing the integer order time derivatives by those of fractional order (Gaul, Klein and Kempfle 1991).

The fractional derivatives appear to be complicated in the time domain (see, Grünwald 1867 or Oldham and Spanier 1974), however, the Laplace transformed representations are defined in a very simple form

$$\mathscr{L}\left\{\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha\mathscr{L}\{x(t)\} \ , \tag{37}$$

where α is of fractional order, and vanishing initial conditions are assumed.

Introducing these fractional derivatives, the viscoelastic constitutive Eq. (36) are generalized to

$$\sum_{k=0}^{M} a_k\frac{\mathrm{d}^{\alpha_k}}{\mathrm{d}t^{\alpha_k}}s_{ij} = \sum_{k=0}^{M} b_k\frac{\mathrm{d}^{\beta_k}}{\mathrm{d}t^{\beta_k}}e_{ij} \text{ and } \sum_{k=0}^{M} c_k\frac{\mathrm{d}^{\alpha_k}}{\mathrm{d}t^{\alpha_k}}\sigma_{ii} = \sum_{k=0}^{M} d_k\frac{\mathrm{d}^{\beta_k}}{\mathrm{d}t^{\beta_k}}\varepsilon_{ii} \ . \tag{38}$$

For $M = 1$, a representation of uniaxial stress-strain relation may be given as a generalized three parameter model by ($a_0 = 1$, $a_1 = p$ and $b_0 = E,\ b_1 = Eq$)

$$p\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\sigma + \sigma = E\left(\varepsilon + q\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\varepsilon\right) \ . \tag{39}$$

According to Bagley and Torvik (1986), a realistic model of the viscoelastic phenomena should be restricted to non-negative internal work and to a non-negative rate of energy dissipation. To satisfy these restrictions, constraints on the model parameters are developed. The finiteness of viscoelastic wave velocities depends on the initial modulus of the relaxation functions. In accordance with actual experimental data it was shown (Gaul and Schanz 1994a) that, if the orders of the fractional derivatives α and β are equal, they are finite. This produces the constraints

$$\begin{aligned} &E \geq 0, \quad q > p \\ &q \geq 0, \quad 0 < \alpha < 2 \\ &p > 0, \quad 0 < \nu < 0.5 \end{aligned} \tag{40}$$

When identical damping mechanisms are assumed for the hydrostatic state and for the deviatoric state, the corresponding 3-D constitutive equations are gained by replacing in the uniaxial stress and strain relations (39) the deviatoric states and the hydrostatic states, respectively

$$\begin{aligned} p\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}s_{ij} + s_{ij} &= \frac{E}{1+\nu}\left(e_{ij} + q\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}e_{ij}\right) \\ p\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\sigma_{kk} + \sigma_{kk} &= \frac{E}{1-2\nu}\left(\varepsilon_{kk} + q\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\varepsilon_{kk}\right) \quad . \end{aligned} \tag{41}$$

Other viscoelastic models with fractional derivatives may be found in the paper of Makris and Constantinou (1993).

3.2.2
Viscoelastic BE formulation

As mentioned above, the viscoelastic BE formulation is achieved by replacing the elastodynamic Laplace transformed fundamental solutions in Eq. (30) and Eq. (31) by viscoelastic ones. It is well known that the elastodynamic fundamental solutions (32) and (33) can be converted to viscoelastic fundamental solutions with the elastic-viscoelastic correspondence principle (Flügge 1975). According to this principle, the viscoelastic fundamental solutions are found from the elastic ones by replacing the elastic moduli in the Fourier or the Laplace transformed domain by the transformed impact response functions of the viscoelastic material model.

For the above mentioned model (41) the elastic-viscoelastic correspondence is given in Laplace domain by

$$\frac{E}{1+\nu} \rightarrow \frac{E}{1+\nu}\frac{1+qs^\alpha}{1+ps^\alpha} \quad \frac{E}{1-2\nu} \rightarrow \frac{E}{1-2\nu}\frac{1+qs^\alpha}{1+ps^\alpha} \tag{42}$$

for the deviatoric part and the hydrostatic part, respectively. In the correspondence (42), the transformation property (37) and the assumption of having the same damping mechanisms in the hydrostatic and deviatoric stress strain state is considered. Furthermore, under this assumption, Poisson's ratio ν is unchanged and, therefore, still a real number. Hence, to convert the elastic fundamental solutions (32) and (33) to viscoelastic ones, only the wave velocities c_1 and c_2 must be replaced as follows:

$$c_1 \rightarrow c_1\sqrt{\frac{1+qs^\alpha}{1+ps^\alpha}} \quad c_2 \rightarrow c_2\sqrt{\frac{1+qs^\alpha}{1+ps^\alpha}} \quad . \tag{43}$$

Then, the obtained viscoelastic fundamental solutions are inserted in the integration weights (30) and (31). The integral free term c_{ij} remains unchanged because it depends only on the geometry and ν. This leads finally to a viscoelastic Boundary Element formulation.

The presented formulation is as singular with respect to r as the elastodynamic formulation. Also, the order of the singularities is identical to that in elastodynamics since only the dependence on time, not on the spatial variable, has been changed. Therefore, also all the other steps of the boundary element methodology, i.e., the evaluation of the integrals, the collocation and the direct equation solving can be transfered from the elastodynamic case.

4
Numerical example

As a first application, a bar ($3m \times 1m \times 1m$) is considered (see, Fig. 2). The bar is taken to be fixed on one end, and is loaded with a unit step function in time on the other free end. The remaining surfaces are traction free. The bar is discretized with 56 triangles, and linear shape functions are used. The material data are: Young's modulus $E = 1\frac{N}{m^2}$, Poisson's ratio $\nu = 0$, density $\varrho = 1\frac{kg}{m^3}$. The parameter τ and L are chosen as suggested in Sect. 2: $L = N$ and $\tau^N = \sqrt{\epsilon}$, with the error bound $\epsilon = 10^{-10}$. Smaller values of ϵ, e.g., below $\epsilon = 10^{-30}$, lead to unstable results.

The spatial integration is done with standard Gauss quadrature formulas. The weakly singular integrals in (30) are regularized by a coordinate transformation and the strongly singular integrals in (31) by the method suggested by Guiggiani and Gigante (1990).

4.1
Elastic results

First, the boundary element formulation (29) with the elastodynamic fundamental solutions (32) and (33) is used. Results for the longitudinal displacement at the point P versus time are plotted for different time step sizes in Fig. 3. There, a backward differential formula of second order is applied for the underlying multistep method. These results are compared with the 1-D solution (Graff

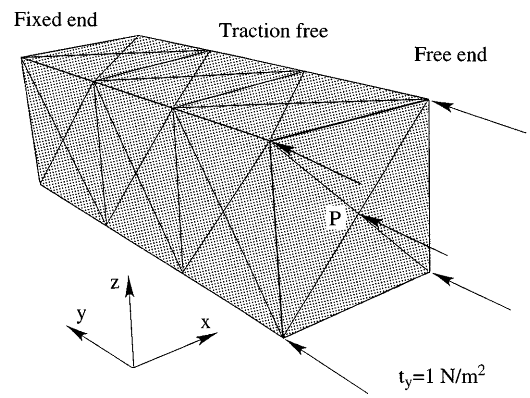

Fig. 2. Discretization, loading and boundary conditions of the bar

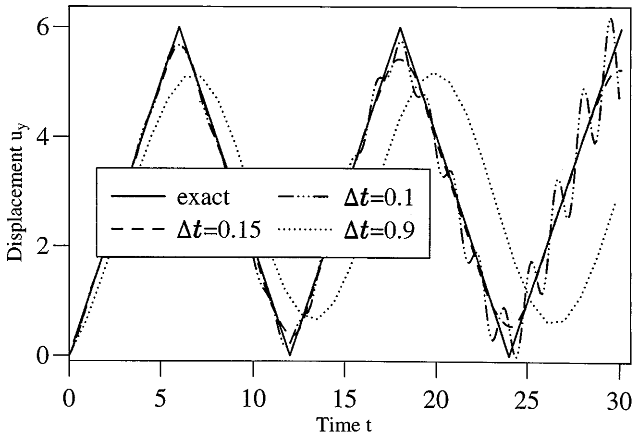

Fig. 3. Longitudinal displacement at point P versus time for different values of Δt

1975), which is denoted with 'exact'. Obviously, there exists a critical value of the time step size Δt below which the results are unstable. For larger time step sizes, a kind of phase shifting is observed because the time step size is too large to approximate the response peaks correctly. But, in comparison with the direct time domain based method suggested by Mansur and Brebbia (1983), the critical value of Δt is smaller. There, the time step size must be chosen between $0.7 < \Delta t < 1$ for this example. For $\Delta t < 0.7$, the method becomes also unstable, while for values of $\Delta t > 1$, the method shows large numerical damping. With the new method, a critical value of $\Delta t < 0.15$ was observed. For very large time step sizes, one obtains a worse approximation of the time history, but no numerical damping.

The existence of a lower critical limit is in accordance with the investigations for the boundary integral equation of the scalar wave equation by Lubich (1994). There, it was proved that the "Operational Quadrature Method" has the same stability as the underlying multistep method. In this study, a Galerkin formulation of the BEM was used, while a collocation method was applied here. Moreover, it was shown that only when performing an approximated spatial integration, a lower limit for a stable time step size Δt exists, i.e., an exact analytical evaluation of all integrals would result in a stable procedure for any, even very small Δt.

The influence of the underlying multistep method is studied in Fig. 4. There, the results with a backward differentiation formula of the first order (BDF 1) and those of the second order (BDF 2) are compared. An optimal choice of the time step size Δt is used in both calculations. The results of the BDF 2 are closer to the 1-D solution than the results of BDF 1, but with the BDF 1 procedure much smaller time step sizes are possible.

One fact about the present numerical implementation should be mentioned. In the method suggested by Mansur and Brebbia (1983), the with respect to time analytically integrated fundamental solutions are zero for $n \geq \frac{r_{\max}}{c_2 \Delta t} + 2$,
where $r_{\max}$ is the maximum distance between two collocation points. In the method presented here, a similar phenomenon is observed. Starting with the same time step $n \geq \frac{r_{\max}}{c_2 \Delta t} + 2$, the integrations weights are several decimal powers smaller than for the time steps before, so that these weights can be neglected. Nevertheless, the point, from which on the weights may be neglected, should be a power of 2 in order to have an efficient FFT.

4.2
Viscoelastic results

In a viscoelastic test, the generalized three parameter model (39) is used. The parameters of the viscosity are chosen to be $p = 0.8$, $q = 1$ and $\alpha = 1.3$. The same bar (Fig. 2) as in the elastodynamic test is studied.

It is a reasonable assumption that the behaviour of the viscoelastic formulation is similar to that of the elastodynamic formulation. In Fig. 5, the longitudinal displacements u_y of point P are plotted versus time for different values of Δt. The solution denoted with 'exact' is the 1-D solution which is performed from the elastodynamic one with the elastic-viscoelastic correspondence principle. The accordance of the 'exact' solution with the BE formulation is very good. Moreover, like in the elastodynamic case, a critical value of Δt is observed. However, the limit of the instabilities are not so sharp. The approximation of the curve is also not exact enough when larger values of Δt are taken. For the sake of brevity, the test for different multistep methods is skipped. The results are similar to those in the elastodynamic case.

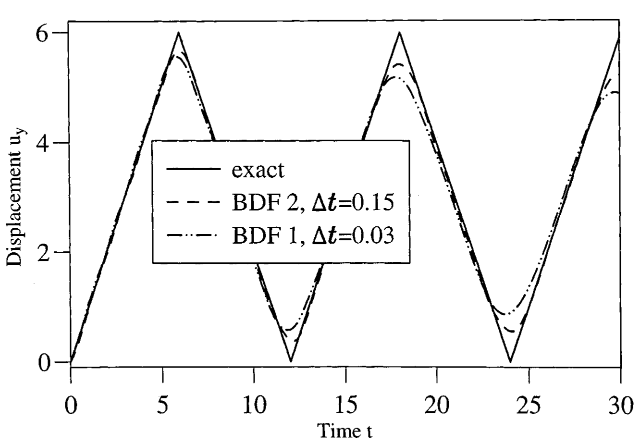

Fig. 4. Comparison of different multistep methods

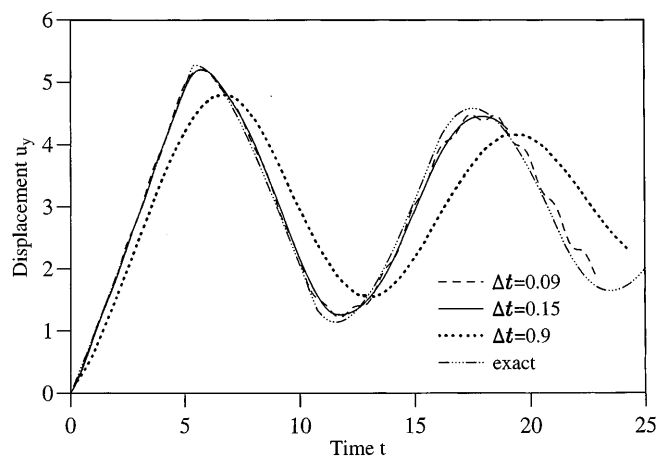

Fig. 5. Viscoelastic response for different Δt

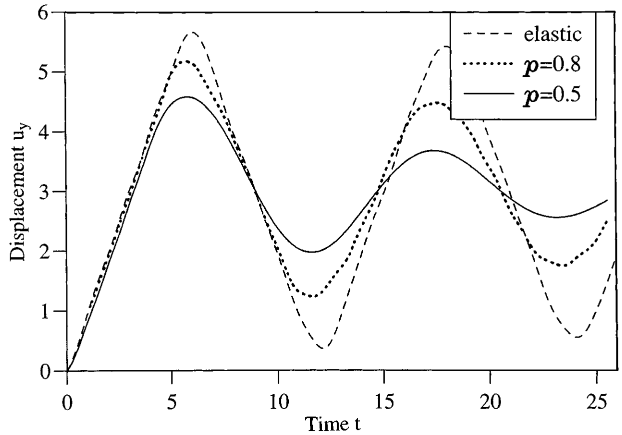

Fig. 6. Viscoelastic response for different p

To show the influence of the viscosity, in Fig. 6, the longitudinal displacement at point P is plotted versus time for different values of p, with constant q and α. For smaller values p, i.e., the ratio $\frac{q}{p}$ becomes larger, the damping increases. The viscoelastic wave velocities, defined with the initial moduli (Gaul and Schanz 1994a),

$$c_{1v} = c_1 \sqrt{\frac{q}{p}} \quad \text{and } c_{2v} = c_2 \sqrt{\frac{q}{p}} \tag{44}$$

also increase, because the material is stiffened. This is also observed in Fig. 6.

Furthermore, the effect of the fractional order α is of interest. In Fig. 7, the displacement is plotted for different values of α for constant p and q. It is observed that higher fractional orders increase the influence of viscosity.

5 Conclusions

The present paper describes a boundary element formulation directly in time domain where only the fundamental solutions of the Laplace domain are used. This formulation is based on the "Operational Quadrature Methods" developed by Lubich (1988a). Applying these quadrature formulas to the convolution integral in the boundary integral equation, a numerical integration formula is obtained where the weights depend only on the Laplace transformed fundamental solutions. Then, after developing a elastodynamic formulation, the elastic-viscoelastic correspondence principle is applied to extend the methodology to a viscoelastic formulation including fractional derivatives in the constitutive equation. A numerical example shows that a critical time step size exists, below which the method becomes unstable. This critical value depends on the underlying multistep method. Compared with the direct time domain based formulation suggested by Mansur and Brebbia (1983), the critical time step size is much smaller. Finally, the effects of the viscosity are shown.

Furthermore, all advantages of the Laplace domain boundary element formulation can be used. Therefore, this method seems to be suitable for all cases where the time-dependent fundamental solution is not known.

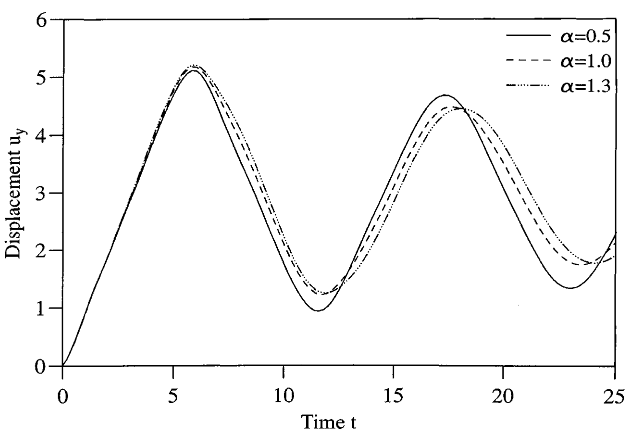

Fig. 7. Viscoelastic response for different α

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