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Dynamical Calculation of Plane Frameworks using Boundary Integral Equations

The beam bending problem according to Timoshenkos theory can be solved using the Finite Element method. Here, the Boundary Integral Method is applied to the bending problem in frequency domain, providing an analytical solution for beams and systems of beams like trusses. An example is presented to demonstrate the method.

1. Timoshenkos equation of bending

Contrary to the well-known Euler/Bernoulli theory of bending, Timoshenkos theory of bending also takes into account the effect of shear deformation and rotatory inertia. The theory can be written in a system of coupled differential equations

$$\mathbf{B}_t \begin{bmatrix} w \\ \varphi_y \end{bmatrix} = \begin{bmatrix} \kappa GA\frac{\partial^2}{\partial x^2} - \rho A\frac{\partial^2}{\partial t^2} & \kappa GA\frac{\partial}{\partial x} \\ -\kappa GA\frac{\partial}{\partial x} & EI_y\frac{\partial^2}{\partial x^2} - \kappa GA - \rho I_y\frac{\partial^2}{\partial t^2} \end{bmatrix} \begin{bmatrix} w \\ \varphi_y \end{bmatrix} = -\begin{bmatrix} q_z \\ m_y \end{bmatrix} \tag{1}$$

in which the variables have the following meaning:

w :	deflection,	φ:	rotation,	EI_y:	flexural rigidity	ρ:	mass
G:	shear modulus	A:	cross sectional area	κ:	shear coefficient	q_z, m_y:	lateral, momentum load .

Performing a Fourier transformation on (1), the time dependence is replaced by a frequency dependence with an excitation frequency ω

$$\mathbf{B}_\omega \begin{bmatrix} \hat{w} \\ \hat{\varphi}_y \end{bmatrix} = \begin{bmatrix} \kappa GA\frac{\partial^2}{\partial x^2} + \rho A\omega^2 & \kappa GA\frac{\partial}{\partial x} \\ -\kappa GA\frac{\partial}{\partial x} & EI_y\frac{\partial^2}{\partial x^2} - \kappa GA + \rho I_y\omega^2 \end{bmatrix} \begin{bmatrix} \hat{w} \\ \hat{\varphi}_y \end{bmatrix} = -\begin{bmatrix} \hat{q}_z \\ \hat{m}_y \end{bmatrix} \quad . \tag{2}$$

In (2), $(\hat{\cdot})$ denotes the amplitude of the corresponding variables.

2. Boundary element equation

The most general methodology to derive from differential equations equivalent integral equations is the method of weighted residuals. The weighted residuum of (2) reads

$$\int_0^\ell \left(\mathbf{B}_\omega \begin{bmatrix} \hat{w}(x) \\ \hat{\varphi}_y(x) \end{bmatrix} + \begin{bmatrix} \hat{q}(x) \\ \hat{m}(x) \end{bmatrix} \right)^T \begin{bmatrix} \hat{w}_q^\infty(x,\xi) & \hat{w}_m^\infty(x,\xi) \\ \hat{\varphi}_q^\infty(x,\xi) & \hat{\varphi}_m^\infty(x,\xi) \end{bmatrix} dx = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad . \tag{3}$$

As weighting function, the matrix of fundamental solutions derived in [1] is chosen. Performing two partial integrations and taking into account the filtering effect of the Dirac operator, the differential operator $\mathbf{B}_\omega$ is shifted from $\hat{w}(x)$ and $\hat{\varphi}(x)$ on the matrix of fundamental solutions. The resulting exact equation reads

$$\begin{aligned} \begin{bmatrix} \hat{w}(\xi) \\ \hat{\varphi}(\xi) \end{bmatrix} &= \int_0^\ell \begin{bmatrix} \hat{w}_q^\infty(x,\xi) & \hat{\varphi}_q^\infty(x,\xi) \\ \hat{w}_m^\infty(x,\xi) & \hat{\varphi}_m^\infty(x,\xi) \end{bmatrix} \begin{bmatrix} \hat{q}(x) \\ \hat{m}(x) \end{bmatrix} dx \\ &+ \left[\begin{bmatrix} \hat{w}_q^\infty(x,\xi) & \hat{\varphi}_q^\infty(x,\xi) \\ \hat{w}_m^\infty(x,\xi) & \hat{\varphi}_m^\infty(x,\xi) \end{bmatrix} \begin{bmatrix} \hat{Q}(x) \\ \hat{M}(x) \end{bmatrix} - \begin{bmatrix} \hat{Q}_q^\infty(x,\xi) & \hat{M}_q^\infty(x,\xi) \\ \hat{Q}_m^\infty(x,\xi) & \hat{M}_m^\infty(x,\xi) \end{bmatrix} \begin{bmatrix} \hat{w}(x) \\ \hat{\varphi}(x) \end{bmatrix} \right]_{x=0}^{x=\ell} \quad . \end{aligned} \tag{4}$$

The above described procedure can similarily be applied to the bar equation in frequency domain (for details on derivation and fundamental solution see [1]). Using both beam and bar equations, point collocation is performed, i.e., $\xi = 0 + \epsilon$ and $\xi = \ell - \epsilon$ with $\epsilon \to 0$, which gives a system of six equations of the form

$$\mathbf{K}^{dyn}\mathbf{d} = \mathbf{r} \quad with \quad \mathbf{d} = \left[\hat{u}(0)\ \hat{w}(0)\ \hat{\varphi}(0)\ \hat{M}(0)\ \hat{Q}(0)\ \hat{N}(0)\ \hat{u}(\ell)\ \hat{w}(\ell)\ \hat{\varphi}(\ell)\ \hat{M}(\ell)\ \hat{Q}(\ell)\ \hat{N}(\ell)\right]^T \tag{5}$$

where $\mathbf{K}^{dyn}$ contains integrals over the fundamental solutions, $\mathbf{d}$ is the vector of unknown boundary values and $\mathbf{r}$ is the loading vector. Since in a single beam problem always six of the twelve state variables at the beam ends are given as boundary conditions, the six equations suffice to solve the problem. After solving the system, values of the state variables at inner points of the beam can be found using (4) and the respective equation for the bar.
When systems of beams like trusses are considered, the element equation (5) can be transformed into global coordinates. Then, taking into account transition conditions at the coupling points of the beams, a system of linear equations for the truss structure can be derived which is easily solved by Gaussian elimination. Contrary to the Finite Element Method, the system equation is not symmectric and also forces and bending moments appear as degrees of freedom.
Note that the derived element equation is exact - therefore, the solution of a truss structure problem is also exact.

3. Example calculation

The prescribed procedure is applied on a two storey framework given in Figure 1. Calculations were performed using the described Boundary Integral Method (BIM) and Finite Element Method (FEM) with two different discretizations. The Finite Elements used here have cubic ansatz functions, for details see [3]. On the right hand side, the

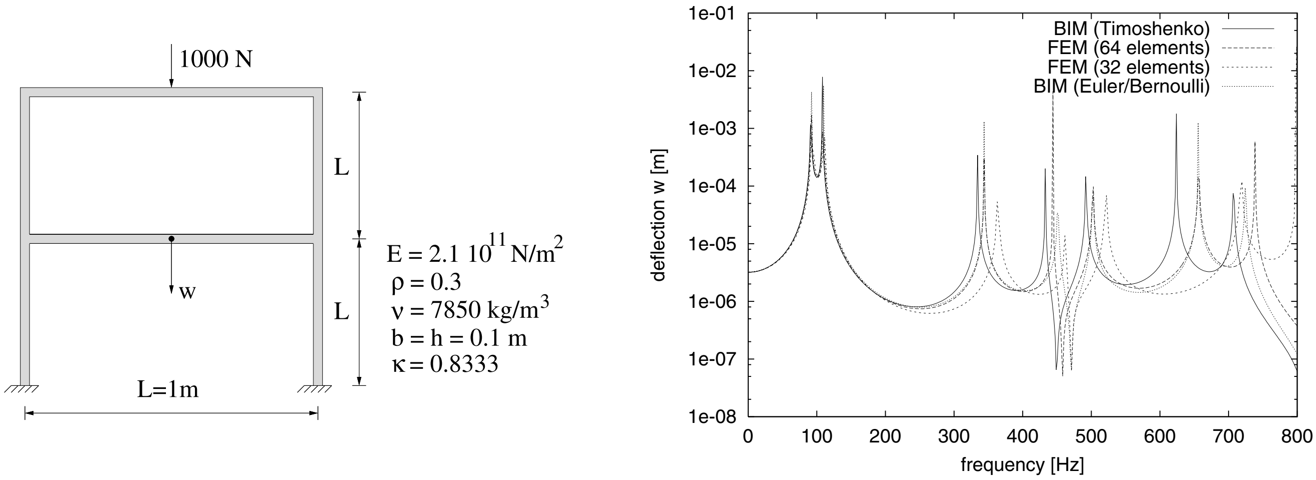

Figure 1: Example calculation in frequency domain: system, loading, results

results for the deflection w of the midpoint of the middle storey is plotted in logarithmic scale against the frequency. For the first modes of vibration, the difference between BIM and FEM results are negligible. For the higher modes, however, one can see that the error of the FEM results increases rapidly.
For comparison, the result of a BEM calculation using Euler/Bernoulli theory is also plotted. The lack of shear deformation and rotatory inertia also causes a considerable shift of the higher modes of vibration.

4. References

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