PHOTOGRAMMETRIC INTERPOLATION *

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ABSTRACT

A number of typical tasks in photogrammetry are seen as interpolation problems. Experiences in some of these are used to advocate a more differentiated judgement of interpolation methods. Some of the methods are compared and an attempt is made to show that it is wise to rely on a number of different interpolation methods for different photogrammetric tasks.

RÉSUMÉ

Un certain nombre de travaux photogrammétriques sont considérés comme étant des problèmes d'interpolation. Les expériences faites dans certains de ces travaux sont utilisées pour préconiser qu'un jugement plus différentié soit apporté entre les diverses méthodes d'interpolation. Quelques-unes de ces méthodes sont comparées et un essai est tenté pour montrer qu'il est judicieux de compter sur plusieurs méthodes d'interpolation pour aborder des différents travaux photogrammétriques.

1 INTRODUCTION

The present paper intends to draw attention to the fact, that interpolation in photogrammetry is too important a problem to approach it with an attitude that "linear interpolation is good enough", that "one method is perfect in all applications", or that "the best method of interpolation is the one which costs the least effort". Instead, in evaluating the problem one has to differentiate many aspects. As an example, superiority of an interpolation algorith does not necessarily have to be based on superior accuracy and/or reduced efforts. It may be based on other characteristics of a method, such as for example smoothing power, or the availability of a quantitative criterion that can replace the intuition of the photogrammetrist in the selection of computational parameters.

The present paper elaborates on these considerations by first dealing with a number of theoretical concepts. These are then supported from specific photogrammetric interpolation experiences, concerning projection errors in Dutch test field photography, film deformation correction, SLAR mapping and interpolation for Digital Terrain Models (DTM)

2 A DEFINITION OF INTERPOLATION

A phenomenon is known at a number of discrete points in n-dimensional space ("reference space"). These points are called "reference-" or "data

*Paper presented at Fédération Internationale des Géomètres, XIV Congress Washington DC September 7–16 1974 points". Interpolation consists of estimating the same phenomenon at intermediate points using the given data. The phenomenon under consideration may be described by a skalar, or by a vector of dimension $m > 1$ (see figure 1a).

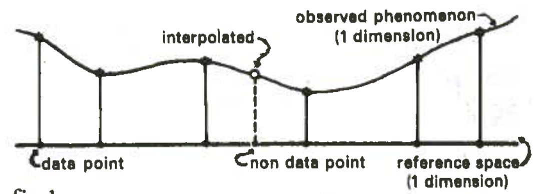


fig 1a

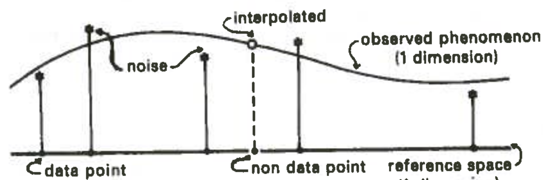


fig 1b

figure 1

This definition is rather practical and does not specifically refer to "smoothing", "filtering" or "regression". These concepts will become relevant, if the data in the reference points are measured, and thus composed of the "signal" and an uncorrelated measuring error, "noise". In this case it becomes a meaningful problem to separate in the data points the observational errors from the signal, and to obtain at non-data points estimates of the signal only ("filtering of the noise" or smoothing (see figure 1b)). And finally, noise does not always have to consist of observational errors, but can simply represent an uncorrelated component in the data much like observational errors; in this case, the signal is the correlated part of the data.

Interpolation as defined here is part of the more general mathematical theory of approximation, of which it represents a particular application. Further pertinent terminology refers to "curve-" and "surface fitting" and "prediction" (to denote interpolation and extrapolation).

In the following these concepts will be explained by a number of tasks in practical photogrammetry. In this context, the term "interpolation" is also used to denote problems covering smoothing, filtering, or surface fitting. Although such nomenclature might not satisfy other fields of science, it is the one traditionally understood in photogrammetry [0, 15].

3 PHOTOGRAMMETRIC INTERPOLATION TASKS

Table 1 summarizes a number of photogrammetric tasks which involve interpolation. It also specifies the dimension of the reference space as well as observed phenomena. It turns out that the correction of radial symmetric lens distortion is one of the simplest interpolation problems. The reference space is one dimensional: the radial distance. The observed phenomenon is the one-dimensional radial lens distortion. A more complicated task is interpolation of Δx, Δy, Δz block- or strip-deformation in the 3d co-ordinate system of the strip, block or terrain.

In a number of cases the phenomenon to be studied is observed directly, as for example in lens distortion and film deformation. In others the observations are indirect, as in strip- and block deformation. Typically these indirect observations are obtained as residuals after transformation by the method of least squares in which the mathematical model is imperfect. In this context it is relevant to note that transformation and interpolation are tasks performed sequentially. They can to some extent substitute each other. In the example of absolute orientation of a well controlled photogrammetric model, a conformal transformation followed by an interpolative correction of model deformation might be as effective as a sole transformation with more than just the conformal parameters.

Moreover, a perfect mathematical model for a least squares adjustment will result in purely uncorrelated residuals. If however the mathematical model is simplified, there will then be a "signal" left in the residuals, so that post-processing of the least squares adjustment results can be useful. Least squares adjustment, filtering and interpolation can be combined in one algorithm. This is called by **Moritz** "least squares collocation".

TASK	DIMENSIONS OF REFERENCE SPACE	DIMENSION OF PHENOMENON
determination of refractive index	3 (x, y, z co-ordinates, plus event. time?)	1 (refractive index)
lens distortion correction in photograph (a)	2 (x, y image co-ordinates; or radial distance r and azimuth)	2 (tangential and radial distortion; or Δx, Δy image errors)
lens distortion correction in photograph, only radial (b)	1 (radial distance r)	1(radial distortion)
film deformation correction	2 (x and y image co-ordinates)	2 (Δx, Δy film deformations)
rectification	2 (x and y image co-ordinates)	2 (Δx, Δy image deformations)
instrumental error correction (a) comparator (b) plotter	 2 (x, y co-ordinates) 3 (x, y, z model co-ord.)	 2 (Δx, Δy) 3 (Δx, Δy, Δz)
model deformation correction	3 (x, y, z model co-ord.)	3 (Δx, Δy, Δz model deformations)
external strip adjustment, planimetry and height	3 (x, y, z strip co-ordinates)	3 (planimetry: Δx, Δy; & height: Δz)
external block adjustment planimetry + height	3 (x, y, z block co-ordinates)	3 (planimetric + height deformations)
digital terrain model (DTM)	2 (x, y reference plane)	1 (z ... height)

table 1: photogrammetric tasks involving interpolation

4 A CLASSIFICATION OF INTERPOLATION METHODS

Interpolation methods could be classified according to the purpose for which they would be suited, eg according to **Rice**, whether they are:

for mathematical representation (derive values at non-data points);

for data analysis (smoothing, extract signal, analyse trend);

for data compression (elimination or redundant information);

for easy manipulation and evaluation

Photogrammetric interpolation tasks often combine some or all of these four objectives. In such a case another classification might be appropriate, depending on whether:

interpolation is with a single, global function;

interpolation is by piecewise, locally defined functions;

interpolation is pointwise.

Interpolation with a global function is applied for example in strip adjustment. All data points are used simultaneously to define the interpolating function. This might be acceptable for strip adjustment with only a few control points.

But for a large number of control points a low order function cannot conform to all data points. High order functions tend on the other hand to be unstable if an orthogonalization procedure is not used. A solution to this dilemma is interpolation by piecewise functions [6]. The reference space is subdivided into patches and for each patch another interpolation function is defined. Often the necessity arises to enforce some continuity between neighbouring patches to avoid cracks. This does not necessarily require explicit consideration of boundary or joining conditions [2] . Typical piecewise interpolation is by linear interpolation, piecewise polynomial interpolation, double linear interpolation [13] , or spline functions.

Pointwise interpolation defines a new interpolation function for each non-data point, using the surrounding subset of data points. Pointwise interpolation is flexible and does not require extensive computer memory;

but it is often slower than the other two classes of interpolation. Typical pointwise interpolation methods are by moving averages [3], [14], linear prediction [5] and weighted arithmetic mean. A detailed description of each of these can be found in the references and a short review is included in the appendix.

Instead of referring to "methods of interpolation" **Rice** refers to "algorithms". With these he denotes a computer programme performing an interpolation task. This algorithm, however, is composed of different constituents, namely:

interpolation form (polynomials, piecewise functions, etc);

error measure (least squares, perfect fit at data points, etc);

method of solving for the unknown.

Rice mentions that the same constituents can produce a number of alternative algorithms and compares the situation with "cooking".

5 CONCEPTS FOR EVALUATION OF INTERPOLATION METHODS

5.1 GENERAL

At present there is a need for objective comparison of interpolation methods in photogrammetry. In the past, the parameters of a specific method have usually been optimised for a specific application. One of the reasons for this might be the degree of intuition often used in choosing an interpolation method, and a lack of criteria over and above accuracy for differentiating between methods. A common approach is therefore to assert that all interpolation methods work equally well, so that the simplest (and cheapest) can be used.

But it has been experienced that there are cases where significant differences exist in the performance of different interpolation methods, even with respect to overall accuracy. Whereas in other cases this accuracy might not differ as between one method and another. The performance of an interpolation algorithm may vary considerably as a function of the structure of the input data (distribution of data points).

In addition to having performance criteria other than accuracy, the evaluation of interpolation methods is made difficult by this dependence on input data structure. But for a given input, **Rice** identifies a series of properties of interpolation algorithms to be used for comparative evaluation.

speed (of solving for unknowns)

flexibility (overall accuracy and maintaining shape of small features)

smoothing power

constraint imposition (terrain break lines in DTM)

memory requirement

smoothness (continuous, derivatives)

speed of evaluation (of interpolating function)

Speed of solving for the unknowns and speed of evaluation cannot be considered separately in pointwise interpolation, but can be separated in piecewise polynomials. Smoothing power refers to the capability of filtering a measuring error from the given data, whereas smoothness refers to the appearance of an interpolated curve or surface, and whether it is continuous or not. From practical experience it seems useful to add to these properties

usefulness for extrapolation

and

reliability (sensitivity to right choice of parameters).

5.2 THE ACCURACY OF INTERPOLATION METHODS

The evaluation of interpolation methods is usually attempted on a basis of their accuracy, which can be described by a root mean square interpolation error. What is this error and how can it be obtained?

In a controlled experiment, the interpolation error can be found by using checkpoints in which the interpolated and the known values are compared. Such an error is composed of the propagation of the measuring error into the interpolated value, and of the loss of information through sampling of the phenomenon at discrete points. In actual applications, the interpolation error can be estimated by interpolation in data points, without using the information of the data point except for comparison with the interpolated value. If sampling was regular, then this method might produce an error estimate which is too large, since the distance between any non-data point to the closest reference value is at least half as small as the distance from such check points to the closest data point. The error estimate can however be used for a comparative evaluation of different interpolation methods, or the optimisation of parameters within a method [8]. Propagation of variances into an interpolated value does not result in an estimate of the interpolation error. Such propagation would only account for the effects of measuring errors (noise) which will often be much smaller than the effect of a limited sampling density.

An approach to evaluating the accuracy of interpolation can be derived from the theory of random functions [1], assuming that a phenomenon can be defined as a stochastic function $z = f(x)$, which is characterized by the covariance function cov (x_i, x_j), between the random variables $z_i = f(x_i)$, $z_j = f(x_j)$; $x_i, x_j \epsilon x$, where x is the continuous range of definition of the independent variable. All interpolation methods mentioned in section 4 can be described as a linear relation between the interpolated value z_p and the values in the data points $z_1, z_2 \ldots z_n$:

$$\bar{z}_p = a_1 \cdot z_1 + a_2 \cdot z_2 + \ldots + a_n \cdot z_n = \underline{a}^t \cdot \underline{z}$$

The coefficients $\underline{a}$ are specific for each interpolation method (and also specific for each position of the non-data point). Variance propagation is applied to the expression

$$\epsilon = z_p - \bar{z}_p$$

with the result that

$$\sigma^2_{\bar{Z}} = \text{cov}(x_p, x_p) + \underline{a}^t \cdot \underline{\text{Cov}}(x_i, x_j) \cdot \underline{a} - 2.\underline{a}^t.\, \underline{\text{cov}}(x_p, x_j)$$

Here, $\text{cov}(x_p, x_p)$ is a skalar, namely the variance of the phenomenon.

$\underline{\text{Cov}}(x_i, x_j)$ is a matrix of covariances among all n data points $(i = 1, \ldots.. n;\ j = 1, \ldots.. n)$.

$\underline{\text{cov}}\,(x_p, x_i)$ is a vector of covariances between the interpolated and given values. The elements of this vector depend on the position of the non-data point.

Clerici and **Kubik** [1] have used $\sigma^2_{\bar{Z}}$ for a comparative evaluation of linear prediction and linear interpolation. The conclusion was that there is hardly a significant difference between the accuracy of the two methods. This conclusion is, however, based on the assumption that the phenomenon can fully be described by the covariance function. Often this might not be the case. This might explain why, in contradistinction to the conclusion in [1], significant differences between linear interpolation and other methods were encountered in an experiment with DTM data [9].

5.3 PROPERTIES OF SOME INTERPOLATION METHODS

The judgement of interpolation methods should consider the structure of the input data to which the methods are to be applied. A thorough effort in this direction should be undertaken since, at the present time, only a subjective judgement for square-grid interpolation can be presented in table 2. Its main purpose is to give an example of an attempt to evaluate interpolation methods. It is intended as a basis for discussion only and not an authoritative classification.

	linear interpolation	polynomial	meshwise bi-linear polynomial	double linear interpolation	weighted arithmetic mean	pointwise linear prediction	zonewise linear prediction	moving average	piecewise polynomial
speed	F	F	F	F	F	U	M	U	U
accuracy	U	U	M	M	U	F	F	F	F
smoothing power	U	M	U	U	M	F	F	F	M
constraint imposition	U	M	U	U	M	F	F	F	M
memory requirement	F	F	F	F	F	M	U	F	F
smoothness	M	F	M	M	M	F	F	F	F
speed of evaluation	F	F	F	F	F				F
use for extrapolation	U	U	U	U	F	F	F	U	U

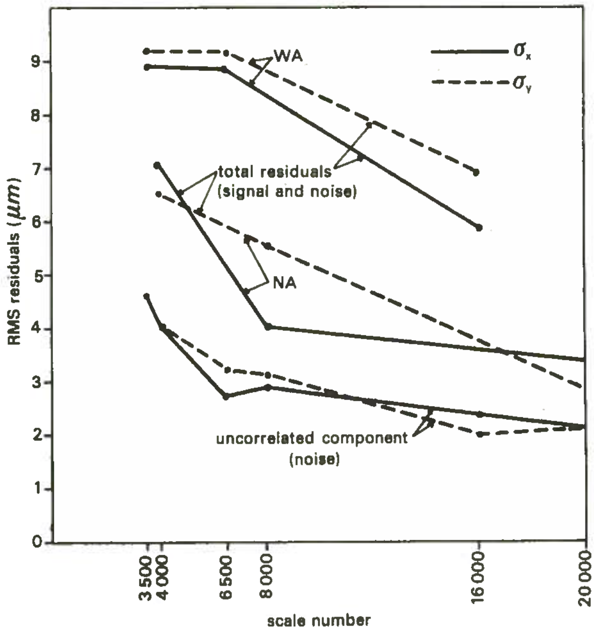
table 2: subjective evaluation of a number of interpolation methods: F ..... favourable; M ..... medium; U ..... unfavourable ; square-grid interpolation assumed

6 SOME EXPERIENCES WITH INTERPOLATION PROBLEMS

6.1 FILTERING OF PROJECTION ERRORS IN TEST FIELD PHOTOGRAPHY

On behalf of the Dutch Photogrammetric Society a large set of aerial photographs was obtained of the Flevopolder test field, with different cameras and at different flying heights. The purpose was to study the errors of the central projection, comparing photogrammetric points with the "theoretical" points derived from terrestrial surveys by a resection in space. Consequently a set of projection errors was obtained for each of the given photographs, representing a 2 dimensional phenomenon on a 2 dimensional reference space. Part of the data analysis was to study the trend and the amount of random noise in these errors. The details of the project have still to be published, but what is of interest here is the interpolation aspect of it; the usefulness of 4 different algorithms was compared for the purpose. Since one objective was data analysis, the use of a number of methods without smoothing power was not possible (eg linear interpolation). So the methods selected were: a single regression polynomial with 1 to 10 coefficients, independent for Δx and Δy errors; a meshwise 3rd order polynomial according to [2]; a moving average of order 1 to 10; and linear prediction.

It was soon found that for the project undertaken, these methods all produced the same results, which are shown in figure 2. A similar result was obtained by **Kupfer** in comparing a single polynomial with linear prediction in an application of test photography of the Rheidt test area near Bonn, in Germany; a 3rd order polynomial with 10 coefficients was found to be sufficient to describe the signal in the data. "Sufficient" was defined by a lack of correlation in the residuals left after filtering.



separation of projection errors into signal and noise

figure 2

The conclusion one might be tempted to draw from these results is that there are no differences between the accuracy or power of interpolation methods. It will be demonstrated in another experience that such a conslusion is premature. One can only state that the specific data do not show such a difference.

The interpolation aspect of the study demonstrates the importance of the concept of filtering in photogrammetry. By means of the correlation function, the presence of a signal in the data can be verified. In the particular case of the analysis of projection errors, one might use a "signal" found in the data analysis step for correction of projection errors in future flight missions. Such an objective would mean mathematical representation. Pointwise interpolation algorithms are inappropriate for this purpose. Instead of such a pointwise numerical trend, a mathematical global function can be used. Since the global polynomial proved sufficient in the data analysis step, it would be the obvious function to use for correction of projection errors.

6.2 PLANIMETRIC MAPPING WITH SIDE-LOOKING RADAR IMAGERY

As part of a large mapping project in Colombia, it was necessary to perform a planimetric triangulation with SLAR imagery covering approximately 400,000 km^2 and 41 ground control points [10]. The task was split into an internal and external adjustment. First, SLAR strips were transformed into a common block system using piecewise 3rd order polynomials with continuous 1st derivative. The internally adjusted block of SLAR images was then transformed into the set of ground control points. Discrepancies in these points were used to compute corrections in radargrammetric points (external adjustment).

The corrections were interpolated by global polynomials, pointwise linear prediction, arithmetic mean and moving averages of order 1 and 3. The results are shown in table 3 and are self-explanatory.

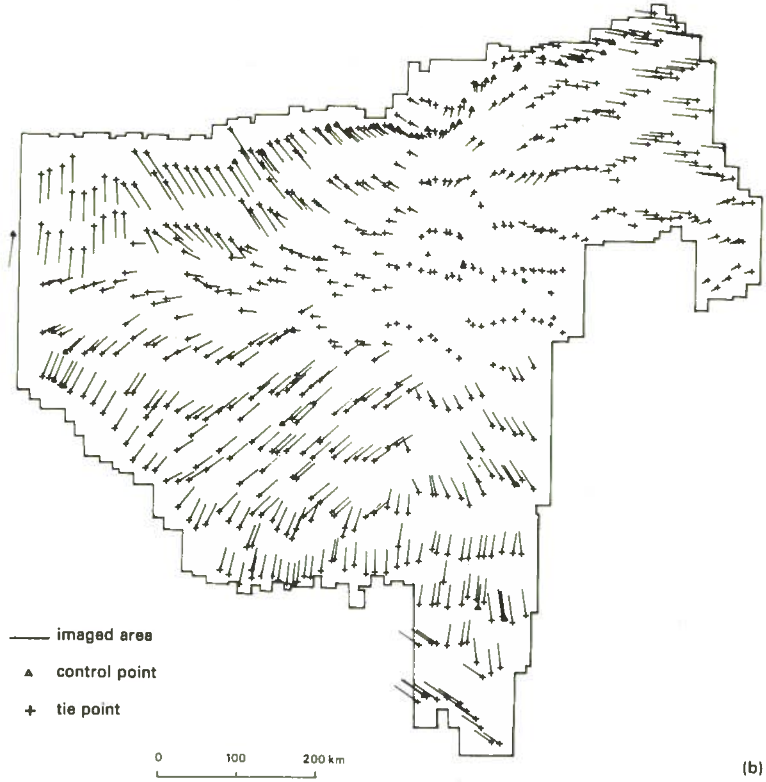
		after 4–parameter fit of SLAR block into all control points	linear prediction, with a 10% filter	moving average, weights equal, order 1	moving average, equal weights, order 3	arithmetic mean weight $1/d^2$	regression polynomial 6 coefficients	regression polynomial 10 coefficients
residuals in control points	RMSE X	3.93	0.55	3.69	0.42	0.50	1.42	0.83
	RMSE Y	3.50	0.69	2.48	0.58	0.75	1.58	1.00
corrections in non–data points	RMSE X		3.58			3.34	5.76	4.66
	RMSE Y		3.25			3.00	3.92	7.18

table 3: results of interpolative correction of SLAR block deformations in x and y, using a number of different methods; the values in mm at image scale, computed from 41 control- and 610 tiepoints

The lack of check points prohibited a reliable estimate of the accuracy. But interpolation in each control point, without using it in the computation, leads to an upper bound for the residual errors in non-data points (RMSE X = ± 2.15 mm; RMSE Y = ± 2.14 mm). However, this upper bound is far from the actual accuracy, since control spacing is very large.

The data point distribution created the problem that in some areas extrapolation had to be done rather than interpolation. Figure 3 shows clearly how a 3rd order polynomial degenerates in areas of no control, while linear prediction produces corrections of the order of magnitude of the discrepancies in the control points. This suggested that, for the given project, it had to be an interpolation method such as linear prediction, rather than global polynomials.

figure 3 a
PRORADAM radar mapping of Amazonas Colombia; interpolated corrections of block deformation of SLAR imagery; result obtained with a 10-parameter polynomial



result obtained from linear prediction;
extrapolation occurred in NE and SW

figure 3 b

In conclusion, an interpolation which might turn into extrapolation requires great care in selecting the algorithm. If no check points are available as in PRORADAM, the danger of extrapolation remains hidden. To be safe, linear prediction can always be used in this case. It has the property of producing 0-corrections in areas of no control. Similar to this is the arithmetic mean, though often less accurate and without quantitative filter control. A global function can only be used for interpolation with large areas of no control, if constraints can be imposed on the function pulling it towards zero in areas of no control.

6.3 CORRECTION OF FILM DEFORMATION WITH RESEAU [4, 16, 17]

The importance of filtering the measuring error, or at least an uncorrelated component of the observations, is demonstrated by the results obtained from two methods of correcting film deformation with reseau photography. The observed co-ordinates of a reseau are compared with the theoretical co-ordinates. The differences are the sum of film deformation and the measuring error, and represent the data points for the interpolation of corrections in photogrammetric points.

One method of correcting film deformation in non-data points was based on the assumption that it is equal in both a non-data point, and the nearest reseau point, which in all cases is a distance smaller than 7 mm in reseau photography. To reduce the effect of systematic measuring errors, the nearest reseau point was measured always after measurement at a photogrammetric (non-data) point.

The other method of film deformation correction was by linear prediction, using exactly the same observations as before. Table 4 shows the results of the experiments after relative and absolute orientation: the photogrammetric model co-ordinates are compared with accurate terrestrial points. The root mean square co-ordinate differences without film deformation correction, and with corrections using these two methods, are shown in the table. It is demonstrated that linear prediction produces significantly smaller root mean square differences than the simpler method, which does not allow for any smoothing of observed data.

	without reseau reference [16] model 1	model 2	with reseau, no filtering reference [17] model 1	model 2	with reseau, linear prediction reference [4] model 1	model 2	with reseau, linear prediction reference [17] model 1	model 2
RMSE X	7.8	7.2	8.8	8.0	6.2	6.8	4.6	6.7
RMSE Y	7.9	9.1	8.0	7.0	5.8	5.5	5.5	6.0
RMSE Z	15.8	11.8	12.2	11.8	11.8	10.1	9.8	10.9

table 4: root mean square discrepancies between 2 photogrammetric stereomodels and terrestrial control, in μm at photoscale, with and without use of reseau; scale 1:10500, 23 control point nests with 3 points each

In table 4, results after linear prediction are shown as obtained by two different authors. The fact that these results are somewhat different indicates that the performance might be considerably altered through the parameters used within an interpolation method (in this example: type of trend, type of correlation function, see [4] and [17]).
In conclusion this interpolation demonstrated that there can be differences between accuracies obtained in one or another algorithm, as opposed to a conclusion drawn from a previous experience.

6.4 INTERPOLATION IN SQUARE GRID DTM

In a controlled numerical experiment to relate the accuracy of a Digital Terrain Model (DTM) with the density of sampling terrain along a regular grid, and with the type of terrain [9], it was also possible to compare a number of interpolation algorithms. The methods were compared in their application to six different types of terrain, 8 different sampling densities, and also with a variation of the number of data points to be used in an interpolation of a new value. Table 5 illustrates the overall results of the comparison of methods. The numbers in this table are obtained as root mean square values of the results of over a million interpolations. Each interpolated value was compared with the "true" terrain height. In order to allow comparison of interpolation methods applied to different terrain and different sampling densities, the interpolation error of any method was always divided by the interpolation error after linear interpolation. Consequently, linear interpolation produces an interpolation error of "100%". Other methods produce in general smaller interpolation errors. The differences in performance amount to 24% in the experiment.

method of interpolation weights		linear interpol	bi-linear polynomial	weighted arithm mean $1/d^4$	moving average e^{-4d^2}	meshwise polynomial $1/d^4$	linear prediction $(1+d^2/4)^{-1}$
no of data points used per interpol	4	1.00	0.89	0.92			0.88
	16			0.97	0.76	0.76	0.76
	36			1.03	0.77		0.76

table 5: relative comparison of interpolation methods, applied to square grid DTM; values are in % relative to linear interpolation, and represent interpolation errors in checkpoints

It is beyond the scope of this paper to go into the details of the study, but some conclusions relevant to the present topic are presented here.

Table 5 shows conclusively that linear interpolation produces larger errors than the more complex algorithms of linear prediction, moving averages or piecewise polynomials. On the other hand, it also indicates that there are two groups of methods which perform differently: the "simple" methods (linear interpolation, bi-linear polynomial, double linear interpolation, weighted arithmetic mean); and the "complex" methods. Within a group, the differences are not distinct.

Table 6 compares the efforts necessary when using these methods. The splitting into 2 main groups also persists there: a slight reduction of the interpolation error can only be obtained with considerable increase of time of computation. It should be noted, however, that linear prediction should not be more expensive than linear interpolation, provided it is also only applied to the 4 closest reference points and not more. But in general it is also clear that the benefit of reduced interpolation errors might often not be worth the extra effort involved. Only for specific purposes it will have to be a "complex" method, which is to be used perhaps not for the interpolation error, but for other properties.

method of interpolation		linear inter pol	bi-linear polynom	arithm mean	moving average	meshwise polynomial	linear prediction
no of points used per interpolation	4	0.04	0.04	0.04			0.04
	16			0.05	0.21	0.25	0.12
	36			0.08	0.43		0.23

table 6: comparison of variable computing time per interpolation of a point; values are seconds, valid for the PDP 11/45 of ITC

The efforts shown refer to data points on a regular square grid. It must be stressed that for irregular distribution of data points, linear interpolation becomes as expensive as linear prediction: a point selection algorithm will require considerable effort to define the 3 closest data points.

DTM interpolation does not necessarily require filtering, if it can be assumed that measuring errors are comparatively small. The possibility of filtering might however be desirable in applications where measuring errors are significant, or where smoothing is essential (generalization).

An important problem in photogrammetric work with DTMs is the consideration of terrain break lines. Typically this is a matter of imposing constraints on the interpolated surface. A number of solutions to this problem exist. They have in common the requirement that a pointwise interpolation algorithm be used.[1] Successful attempts to

1
for example: Assmus, E., Extension of Stuttgart Contour Programme to treating Terrain Break Lines ISP-Comm III Symp. Stuttgart, 1974

consider terrain irregularities in piecewise interpolation are not yet known by the author, although an unsuccessful effort has been mentioned by **Jancaitis** and **Junkins**[2].

7 CONCLUSIONS

The number and importance of photogrammetric tasks which make use of interpolation and filtering, suggest that the theory and methods of interpolation can be a worthy tool for the photogrammetrist. However, approaches to interpolation are often intuitive and not systematic. Fortunately there has been a new stimulus to study of the problems of interpolation and filtering, namely the Digital Terrain Model (DTM). The DTM itself does not pose the largest problems, nor does it require the most advanced theories of interpolation. These may be useful in problems of data analysis (eg filtering) as applied to strip and block adjustment.

Although some applications may require very specific solutions, it has been found that a majority of problems can be treated with the same set of algorithms. An important element of these is the possibility of computing correlation functions. These may show, quantitatively, whether the data contain a correlated component. This is rather important for data analysis and helps to avoid interpolation and filtering being attempted in completely random noise. The set of algorithms should further contain a component for computing regression polynomials. This is required for simple smoothing problems (trend analysis), for mathematical representation, and to preprocess data for linear prediction. This latter method should then be available as a general purpose interpolation and smoothing algorithm. Although it is a computer-intensive method (expensive) it is useful because of the fact that it allows for well controlled smoothing and does not degenerate in cases of extrapolation. As a last algorithm it is recommended to have available piecewise polynomials (spline functions). These might be of use, if a more flexible mathematical representation of a phenomenon is required than is possible with a global polynomial.

2
Jancaitis, J.E. and Junkins, J.L., Personal Communication

With this set of algorithms, a number of photogrammetric interpolation tasks have successfully been carried out at the ITC. Some of these tasks have been described in this paper to justify conclusions on a comparison of interpolation methods. It was shown that interpolation errors varied up to 24% using different methods of interpolation. But it was attempted to evaluate methods of interpolation not only on a basis of interpolation errors, but also according to criteria such as smoothing power, usefulness for extrapolation, etc. It is this spectrum of properties which should be used to decide on the choice of a particular method of interpolation, rather than the traditional considerations concerning only interpolation errors and efforts.

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APPENDIX - Interpolation Methods

The following description of interpolation methods assumes a one-dimensional phenomenon on a two-dimensional reference space.

Linear Interpolation

The 3 closest data points with observed values Z_1, Z_2, Z_3 are used to define a new value Z_p by fitting a plane surface, so that

$$Z_p = a_0 + a_1 . x + a_2 . y \qquad (1)$$

Coefficients a_0, a_1, a_2 can be solved from the 3 data points.

Co-ordinates (x_i , y_i) give the location of a point i in the reference space.

Double Linear Interpolation

The 4 closest data points forming a quadrangle are selected. The 4 points define 2 triangles, which each contain the new point. Two linear interpolations can be carried out, in each triangle one linear interpolation. The arithmetic mean produces Z_p as:

$$Z_p = (Z'_p + Z''_p)/2$$

where Z'_p, Z''_p are produced according to (1).

Bilinear Polynomial

The 4 closest data points define a quadrangle. This allows one to compute Z_p using the bilinear polynomial

$$Z_p = a_0 + a_1 . x + a_2 . y + a_3 . x . y$$

Arithmetic Mean

From n data points, a new value Z_p is found from:

$$Z_p = \sum_{i=1}^{n} (x_i/d_i^{\,k}) \,/\, \sum_{i=1}^{n} 1/d_i^{\,k}$$

Here, $d_i^2 = (x_i - x_p)^2 + (y_i - y_p)^2$, and k is selected according to the intuition of the user. Weight $1/d_i^k$ can also be replaced by other functions.

Moving Average

For each new data point, the n surrounding reference points are selected. The new point is chosen as the origin of planimetric co-ordinates. The absolute term of a polynomial of order m is computed from the n data points, giving each of them a different weight, eg according to distance from the new point. The computed absolute term is the interpolated Z_p, since $x_p = y_p = 0$. The described process is a weighted moving average of order m, using n points.

Linear Prediction

A (polynomial) regression function (trend t(x, y)) is computed from n data points. The residuals only can be input to linear prediction. From the residuals, a correlation function cov (d, α) is computed, or chosen a priori.

The correlation function cov (d, α) describes the dependence of two residuals a distance d apart and defining the direction α. Usually, dependence on α is not assumed, so that one uses cov (d) only. A correlation matrix $\underline{\text{Cov}}$ is defined:

$$\underline{\text{Cov}} = \begin{bmatrix} 1 & \text{cov}(d_{1,2}) & \dots & \text{cov}(d_{1,n}) \\ \text{cov}(d_{2,1}) & 1 & \dots & \text{cov}(d_{2,n}) \\ \cdots & \cdots & \cdots & \cdots \\ \text{cov}(d_{n,1}) & \text{cov}(d_{n,2}) & & 1 \end{bmatrix}$$

Distances d_{ij} are between data points i and j. Also a correlation vector $\underline{\text{cov}}$ is defined between the new and data points:

$$\underline{\text{cov}} = (\text{cov}(d_{1,p}),\ \text{cov}(d_{2,p}),\ \dots\dots\ \text{cov}(d_{n,p}))$$

The new point obtains:

$$Z_p = t(x_p, y_p) + \underline{\text{cov}} \cdot \underline{\text{Cov}}^{-1} . \Delta \underline{Z}^t ; \Delta \underline{Z} = (\Delta Z_1 \Delta Z_2 \dots)$$

Careless application of linear prediction can be detrimental and so careful study of the literature (eg [5], [11]) should precede actual use of the method.

Patchwise Polynomials (Spline Functions)

There are many ways of computing patchwise polynomials. Within the interpolation area, a not-necessarily-regular grid is chosen. Within each mesh of the grid, a different polynomial is defined. If the polynomials of order n join along the boundaries of adjacent meshes, with all derivatives up to order n–1 being continuous, one speaks of "spline functions".

Continuity can be obtained by interpolating values and eventually also tangents of the phenomenon in the grid points. This represents a simple, fast, memory saving process, and can be done with a moving average. Next, the generated function values (and tangents) are used to define the polynomial in each mesh. Making an appropriate choice of the polynomials, and having sufficient values at the grid points, the generated polynomials will have continuous (n–1)st, ... 2nd, 1st, 0th derivative.

The method evaluated in section 6.4, applied 3rd order polynomials with 12 coefficients. In each grid point 1 function value and 2 tangents (t_x, t_y) were computed. These were per mesh the 12 given data to define the 12 coefficients of the polynomial piece.

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