Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 124 (2014) 3441–3462

stochastic processes and their applications

www.elsevier.com/locate/spa

# Trimmed stable AR(1) processes

Alina Bazarova [a], István Berkes [a,*], Lajos Horváth [b]

[a] *Institute of Statistics, Graz University of Technology, Kopernikusgasse 24/III, Graz, Austria*
[b] *Department of Mathematics, University of Utah, Salt Lake City, UT 84112–0090, USA*

Received 6 September 2012; received in revised form 2 May 2014; accepted 2 May 2014
Available online 10 May 2014

**Abstract**

In this paper we investigate the distribution of trimmed sums of dependent observations with heavy tails. We consider the case of autoregressive processes of order one with independent innovations in the domain of attraction of a stable law. We show if the $d$ largest (in magnitude) terms are removed from the sample, then the sum of the remaining elements satisfies a functional central limit theorem with random centering provided $d = d(n) \geq n^{\gamma}$ (for some $\gamma > 0$) and $d(n)/n \to 0$. This result is used to get asymptotics for the widely used CUSUM process in case of dependent heavy tailed observations.


*MSC:* primary 60F17; secondary 62M10; 62G10

*Keywords:* Trimming; Heavy tails; Asymptotic normality; Autoregressive(1) processes; CUSUM processes

## 1. Introduction

Let $X_1, X_2, \ldots,$ be independent, identically distributed random variables in the domain of attraction of a stable law with index $0 < \alpha < 2$. Lévy [35] and Darling [19] noted that the order of magnitude of the sum $S_n = \sum_{k=1}^{n} X_k$ is the same as that of its largest term and the contribution of a fixed, but large number of extremal terms is essentially responsible for the distribution of $S_n$.

* Corresponding author. Tel.: +43 316 8736476.
*E-mail addresses:* bazarova@tugraz.at (A. Bazarova), berkes@tugraz.at, berkes@renyi.hu (I. Berkes), horvath@math.utah.edu (L. Horváth).

http://dx.doi.org/10.1016/j.spa.2014.05.001

The asymptotic distribution of the trimmed sum $S_n^{(d)}$ obtained from $S_n$ by discarding the $d$ smallest and $d$ largest summands was determined by LePage et al. [34] and Csörgő et al. [18] proved that in case of moderate trimming, i.e. $d(n) \to \infty, d(n)/n \to 0$ the trimmed sum $S_n^{(d)}$ satisfies the central limit theorem. Arov and Bobrov [2], Mori [40], Hall [30], Teugels [44], Griffin and Pruitt [28,29] and Kesten [32] considered a different type of trimming of the sample. Let $\eta_{n,d}$ denote the $d$th largest element of $|X_1|, \ldots, |X_n|$. These authors were interested in the asymptotic behavior of the modulus trimmed sum ${}^{(d)}S_n = \sum_{k=1}^n X_k I\{|X_k| \le \eta_{n,d}\}$, i.e. when from the sum we remove the $d$ elements with the largest absolute values. Griffin and Pruitt [28] proved that the trimmed central limit theorem of Csörgő et al. [18] remains valid for modulus trimmed sums provided the distribution of $X_1$ is symmetric, but it generally fails for nonsymmetric variables and it can happen that ${}^{(d)}S_n$ is asymptotically normal for some $d(n)$, but not for another $d'(n) \ge d(n)$. This is somewhat unexpected, since removing more large elements from the sample should result in better behavior. Sufficient conditions for the asymptotic normality of ${}^{(d)}S_n$ in the nonsymmetric case were given in [9]. On the other hand, Berkes et al. [11] showed that if $d(n) \to \infty$, $d(n)/n \to 0$, a functional central limit theorem always holds for ${}^{(d)}S_n$ with a random centering factor. Some of these results are extended in [33] to long range dependent sequences.

Trimming also has important applications in statistics. As an example, we consider the detection of possible changes in the location model

$$X_j = c_j + e_j, \quad 1 \le j \le n,$$

where $e_1, \ldots, e_n$ are random errors. Under the null hypothesis

$$H_0: \; c_1 = c_2 = \cdots = c_n$$

of no change in the location parameter we have

$$X_j = c + e_j, \quad 1 \le j \le n, \tag{1.1}$$

with some constant $c$. Under the alternative there are $r$ changes:

$$\begin{aligned} H_A: \quad & \text{there is } r \ge 1 \text{ and } 1 < k_1 < k_2 < \cdots < k_r < n \text{ such that} \\ & c_1 = \cdots = c_{k_1-1} \ne c_{k_1} = c_{k_1+1} = \cdots = c_{k_2-1} \ne c_{k_2} = c_{k_2+1} = \cdots \\ & = c_{k_r-1} \ne c_{k_r} = \cdots = c_n. \end{aligned}$$

The most popular methods to test $H_0$ against $H_A$ (cf. [4,17]) are based on the CUSUM process

$$U_n(x) = \sum_{i=1}^{\lfloor nx \rfloor} X_i - \frac{\lfloor nx \rfloor}{n} \sum_{i=1}^{n} X_i, \tag{1.2}$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Clearly, if $H_0$ is true, then $U_n(t)$ does not depend on the common but unknown location parameter $c_1$. It is well known if $X_1, \ldots, X_n$ are independent and identically distributed random variables with a finite second moment, then

$$\frac{1}{(n\mathrm{var}(X_1))^{1/2}} U_n(x) \xrightarrow{\mathcal{D}[0,1]} B(x),$$

where $B(x)$ is a Brownian bridge and $\xrightarrow{\mathcal{D}[0,1]}$ means weak convergence in the space $\mathcal{D}[0,1]$ of càdlàg functions equipped with the Skorokhod $J_1$ topology (cf. [12]). Assuming that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables in the domain of attraction of a stable law of index $\alpha \in (0, 2)$, Aue et al. [3] showed that

$$\frac{1}{n^{1/\alpha}\hat{L}(n)} U_n(x) \xrightarrow{\mathcal{D}[0,1]} B_\alpha(x),$$

where $\hat{L}$ is a slowly varying function at $\infty$ and $B_\alpha(x)$ is an $\alpha$-stable bridge. (The $\alpha$-stable bridge is defined as $B_\alpha(x) = W_\alpha(x) - xW_\alpha(1)$, where $W_\alpha$ is a Lévy $\alpha$-stable motion.) Since nothing is known on the distributions of the functionals of $\alpha$-stable bridges, Berkes et al. [11] suggested the trimmed CUSUM process

$$T_{n,d}(x) = \sum_{i=1}^{\lfloor nx \rfloor} X_i I\{|X_i| \le \eta_{n,d}\} - \frac{\lfloor nx \rfloor}{n} \sum_{i=1}^{n} X_i I\{|X_i| \le \eta_{n,d}\}. \tag{1.3}$$

Assuming that the $X_i$'s are independent and identically distributed and are in the domain of attraction of a stable law, they proved

$$\frac{1}{\sigma_n} T_{n,d}(x) \xrightarrow{\mathcal{D}[0,1]} B(x), \tag{1.4}$$

where

$$\sigma_n^2 = \frac{\alpha}{2-\alpha} (H^{-1}(d/n))^2 d,$$

$B(t)$ is a Brownian bridge and $H^{-1}$ denotes the generalized inverse of $H$, the survival function of $X_1$. The CUSUM process has also been widely used in case of dependent variables but it is nearly always assumed that the observations have high moments and the dependence in the sequence is weak. For a review we refer to [4]. However, very few papers consider the instability of time series models with heavy tails.

Fama [25] and Mandelbrot [37,38] pointed out that the distributions of commodity and stock returns are often heavy tailed with possibly infinite variance and their research started the investigation of time series models where the marginal distributions have regularly varying tails. Davis and Resnick [21,22] investigated the properties of moving averages with regularly varying tails and obtained non-Gaussian limits for the sample covariances and correlations. Their results were extended to heavy tailed ARCH in [20]. The empirical periodogram was studied by Mikosch et al. [39]. Andrews et al. [1] estimated the parameters of autoregressive processes with stable innovations.

In this paper we extend the results of [11] to a dependent setting. More precisely, we prove a functional central limit theorem for trimmed sums with random centering for an AR(1) process with innovations in the domain of attraction of a stable law. As a consequence we derive a limit result similar to (1.4) for the trimmed AR(1) process.

In Section 2 we formulate our main results. The proofs are based on several technical lemmas, given in Section 3. In Section 4 we prove the main results in the case when trimming is replaced by a (nonrandom) truncation and from these in Section 5 we deduce Theorems 2.1 and 2.2.

## 2. Main results

Let $e_i$ be a non-anticipative (i.e. $\sigma\{\varepsilon_j, j \le i\}$ measurable) solution of

$$e_i = \rho e_{i-1} + \varepsilon_i, \quad -\infty < i < \infty. \tag{2.1}$$

We assume throughout this paper that

$$\varepsilon_j, -\infty < j < \infty \text{ are independent and identically distributed,} \tag{2.2}$$

$$\begin{aligned}&\varepsilon_0 \text{ belongs to the domain of attraction of a stable}\\ &\text{random variable } \xi^{(\alpha)} \text{ with parameter } 0 < \alpha < 2,\end{aligned} \tag{2.3}$$

and

$$\varepsilon_0 \text{ is symmetric when } \alpha = 1. \tag{2.4}$$

Assumption (2.3) means that

$$\left(\sum_{j=1}^{n} \varepsilon_j - a_n\right) \Big/ b_n \xrightarrow{\mathcal{D}} \xi^{(\alpha)} \tag{2.5}$$

for some numerical sequences $a_n$ and $b_n$. The necessary and sufficient condition for this is

$$\lim_{t\to\infty} \frac{P\{\varepsilon_0 > t\}}{L_*(t)t^{-\alpha}} = p \quad \text{and} \quad \lim_{t\to\infty} \frac{P\{\varepsilon_0 \le -t\}}{L_*(t)t^{-\alpha}} = q \tag{2.6}$$

for some numbers $p \ge 0, q \ge 0, p + q = 1$, where $L_*$ is a slowly varying function at $\infty$. It is known that (2.1) has a unique stationary non-anticipative solution if and only if

$$-1 < \rho < 1. \tag{2.7}$$

Under assumptions (2.2)–(2.7), $\{e_j\}$ is a stationary sequence and $E|e_0|^\kappa < \infty$ for all $0 < \kappa < \alpha$ but $E|e_0|^\kappa = \infty$ for all $\kappa > \alpha$. AR(1) processes with stable innovations were considered by Chan and Tran [15], Chan [14], Aue and Horváth [5] and Zhang and Chan [47] who investigated the case when $\rho$ is close to 1 and provided estimates for $\rho$ and the other parameters when the observations do not have finite variances.

The convergence of the finite dimensional distributions of $U_n(x)$ in the AR(1) case is an immediate consequence of Phillips and Solo [42] representation. Let $\xrightarrow{fdd}$ denote the convergence of the finite dimensional distributions. If (1.1)–(2.4) and (2.7) hold, then we have

$$\frac{1-\rho}{n^{1/\alpha}L_*(n)} U_n(x) \xrightarrow{fdd} B_\alpha(x), \tag{2.8}$$

where $B_\alpha(x), 0 \le x \le 1$ is an $\alpha$-stable bridge and $L_*$ is defined in (2.6). It has been pointed out in [6,7] that the *fdd* convergence in (2.8) cannot be replaced with weak convergence in $\mathcal{D}[0, 1]$. However, Avram and Taqqu [7] proved that $U_n(x)$ converges in the weak-$M_1$ sense under some additional regularity conditions. Some of their regularity conditions were removed by Tyran-Kamińska [45]. For further results on the weak convergence of dependent sequences with infinite variance in the $M_1$ topology we refer to [8].

We formulate now our main results. On the truncation parameter $d = d(n)$ we will assume

$$\lim_{n\to\infty} d(n)/n = 0 \tag{2.9}$$

and

$$d(n) \ge n^\delta \quad \text{with some } 0 < \delta < 1. \tag{2.10}$$

Let $F(x) = P\{X_0 \le x\}$, $H(x) = P\{|X_0| > x\}$ and let $H^{-1}(t)$ be the (generalized) inverse of $H$. Our last condition will be used to establish the weak law of large numbers for $\eta_{n,d}$. We assume that $\varepsilon_0$ has a density function $p(t)$ which satisfies

$$\int_{-\infty}^{\infty} |p(t+s) - p(t)|dt \le C|s| \quad \text{with some positive constant } C. \tag{2.11}$$

(Here, and in the sequel, all constants will be finite and positive.) Let

$$A_n = d^{1/2} H^{-1}(d/n) \tag{2.12}$$

and

$$m(t) = EX_1 I\{|X_1| \le t\}. \tag{2.13}$$

**Theorem 2.1.** *If (1.1), (2.1)–(2.4), (2.7) and (2.9)–(2.11) hold, then we have that*

$$\left(\frac{2-\alpha}{\alpha}\right)^{1/2} \left(\frac{1-\rho}{1+\rho}\right)^{1/2} \frac{1}{A_n} \sum_{k=1}^{n} \left[X_k I\{|X_k| \le \eta_{n,d}\} - m(\eta_{n,d})\right] \xrightarrow{\mathcal{D}[0,1]} W(x),$$

*where $W(x)$ is a Wiener process.*

The result in Theorem 2.1 uses the random centering factor $m(\eta_{n,d})$. This factor is characteristic for the asymptotic distribution of the modulus trimmed partial sums process, as first observed in [11]. Since a random translation of the terms in the CUSUM process cancels out, the next result is an immediate consequence of Theorem 2.1.

**Theorem 2.2.** *If (1.1), (2.1)–(2.4), (2.7) and (2.9)–(2.11) hold, then we have that*

$$\left(\frac{2-\alpha}{\alpha}\right)^{1/2} \left(\frac{1-\rho}{1+\rho}\right)^{1/2} \frac{T_{n,d}(x)}{A_n} \xrightarrow{\mathcal{D}[0,1]} B(x),$$

*where $B(x)$ is a Brownian bridge.*

Statistical applications of Theorem 2.2 require the estimation of the norming factor from the observations. We suggest a kernel type estimator for the norming factor in Theorem 2.2 which is computed from the trimmed observations $X_i^* = X_i I\{|X_i| \le \eta_{n,d}\}$. Let

$$\hat{s}_n^2 = \hat{\gamma}_0 + 2\sum_{j=1}^{n-1} \omega(j/h)\hat{\gamma}_j,$$

where

$$\hat{\gamma}_j = \frac{1}{n}\sum_{i=1}^{n-j}(X_i^* - \bar{X}_n^*)(X_{i+j}^* - \bar{X}_n^*) \quad \text{with } \bar{X}_n^* = \frac{1}{n}\sum_{i=1}^{n} X_i^*.$$

For the kernel $\omega(t)$ we assume the following regularity conditions: (i) $\omega(0) = 1$, (ii) $\omega(t) = 0$ if $t > a$ with some $a > 0$, (iii) $\omega$ is Lipschitz continuous, and (iv) the Fourier transform of $\omega$ is Lipschitz continuous and is integrable on the real line. Assuming that $h = h(n) \to \infty$ and $h(n)/n \to 0$, the method in [31,36] can be used to establish that

$$\frac{\hat{s}_n}{A_n} \xrightarrow{P} \left(\frac{\alpha}{2-\alpha}\right)^{1/2} \left(\frac{1+\rho}{1-\rho}\right)^{1/2}.$$

Hence Theorem 2.2 yields

$$\frac{T_{n,d}(x)}{\hat{s}_n} \xrightarrow{\mathcal{D}[0,1]} B(x),$$

where $B(x)$ is a Brownian bridge.

In this paper we considered a stationary AR(1) sequence with stable innovations. Theorems 2.1 and 2.2 could be extended to more general linear processes but this extension would require nontrivial modifications of our method or a completely different approach.

## 3. Preliminary results

The proofs of Theorems 2.1 and 2.2 are based on several technical lemmas.

In the sequel we can and will assume without loss of generality that

$$E\varepsilon_0 = 0, \quad \text{if } 1 < \alpha < 2. \tag{3.1}$$

Under these conditions, in (2.5) we can choose $a_n = 0$ and $b_n$ can be chosen to be any sequence satisfying

$$\frac{n}{b_n^\alpha} L_*(b_n) \to 1. \tag{3.2}$$

According to Theorem 2.3 of Cline [16] (cf. also Davis and Resnick [22]), $H(x)$, the survival function of $|X_0|$, satisfies

$$H(x) = x^{-\alpha} L(x), \tag{3.3}$$

where $L(x)$ is a slowly varying function at $\infty$ and

$$\lim_{x\to\infty} \frac{H(x)}{P\{|\varepsilon_0| > x\}} = \lim_{x\to\infty} \frac{L(x)}{L_*(x)} = \frac{1}{1 - |\rho|^\alpha}. \tag{3.4}$$

Let

$$u_{k,n}(t) = X_k I\{|X_k| \le t H^{-1}(d/n)\} \quad \text{and} \quad m_n(t) = E[X_0 I\{|X_0| \le t H^{-1}(d/n)\}].$$

The main goal of this section is to get bounds for $Eu_{0,n}(t)u_{k,n}(s)$ and $\operatorname{cov}(u_{0,n}(t), u_{k,n}(s))$.

**Lemma 3.1.** *We assume that (1.1), (2.1)–(2.4), (2.7) and (3.1) hold. Let* $\mathbf{Y}^{(k)} = (X_0, X_k)$ *and let* $\mathbf{Y}_i^{(k)}, i = 1, 2, \ldots$ *be independent and identically distributed copies of* $\mathbf{Y}^{(k)}$. *Then*

$$\frac{\mathbf{Y}_1^{(k)} + \cdots + \mathbf{Y}_n^{(k)}}{n^{1/\alpha} L_*(n)} \xrightarrow{\mathcal{D}} \mathbf{Z}^{(k)} \quad \text{as } n \to \infty,$$

*where* $\mathbf{Z}^{(k)} = (Z_1^{(k)}, Z_2^{(k)})$ *with*

$$Z_1^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \xi_{-\ell}^{(\alpha)} \quad \text{and} \quad Z_2^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \xi_{k-\ell}^{(\alpha)}$$

*and* $\xi_\ell^{(\alpha)}, -\infty < \ell < \infty$ *are independent and identically distributed copies of* $\xi^{(\alpha)}$.

**Proof.** It follows from (2.1) that

$$X_k - c = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{k-\ell} = \sum_{\ell=0}^{k-1} \rho^\ell \varepsilon_{k-\ell} + \rho^k X_0, \quad 1 \le k < \infty. \tag{3.5}$$

Let $\varepsilon_\ell^{(i)}, -\infty < \ell < \infty, i = 1, 2, \ldots$ be independent and identically distributed copies of $\varepsilon_0$. Clearly

$$\mathbf{Y}_i^{(k)} = (Y_{i,1}^{(k)}, Y_{i,2}^{(k)}) \quad \text{with } Y_{i,1}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{-\ell}^{(i)} \quad \text{and} \quad Y_{i,2}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{k-\ell}^{(i)}$$

are independent and identically distributed copies of $\mathbf{Y}^{(k)}$. Elementary algebra gives

$$\sum_{i=1}^{n} Y_{i,1}^{(k)} = \sum_{\ell=0}^{\infty} \rho^\ell \sum_{i=1}^{n} \varepsilon_{-\ell}^{(i)} \quad \text{and} \quad \sum_{i=1}^{n} Y_{i,2}^{(k)} = \sum_{\ell=0}^{k-1} \rho^\ell \sum_{i=1}^{n} \varepsilon_{k-\ell}^{(i)} + \rho^k \sum_{\ell=0}^{\infty} \rho^\ell \sum_{i=1}^{n} \varepsilon_{-\ell}^{(i)}.$$

For every $L \ge 0$ by (2.5) we have that (recall that under our conditions the centering factors $a_n$ in (2.5) can be chosen 0)

$$\frac{1}{b_n} \left( \sum_{i=1}^{n} \varepsilon_\ell^{(i)}, -L \le \ell \le L \right) \xrightarrow{\mathcal{D}} \left( \xi_\ell^{(\alpha)}, -L \le \ell \le L \right),$$

where $\xi_\ell^{(\alpha)}, -\infty < \ell < \infty$ are independent and identically distributed copies of $\xi^{(\alpha)}$. Let $0 < \kappa < \alpha$. It follows from Theorem 6.1 of de Acosta and Giné [24, p. 225] that

$$E \left| \frac{1}{b_n} \sum_{i=1}^{n} \varepsilon_\ell^{(i)} \right|^\kappa \le C_1,$$

and therefore for every $x > 0$ we have that

$$\lim_{L\to\infty} \limsup_{n\to\infty} P \left\{ \sum_{\ell=L+1}^{\infty} \rho^\ell \left| \frac{1}{b_n} \sum_{i=1}^{n} \varepsilon_\ell^{(i)} \right| > x \right\} = 0$$

and similarly

$$\lim_{L\to\infty} P \left\{ \sum_{\ell=L+1}^{\infty} \rho^\ell |\xi_\ell^{(\alpha)}| > x \right\} = 0.$$

This completes the proof of the lemma. □

Let $\mathbf{i}$ denote the imaginary unit.

**Lemma 3.2.** *Let* $\mathbf{Y}$ *be a stable vector variable with characteristic function* $\psi(s, t)$. *Then there exists a measure* $\nu$ *on the Borel sets of* $R^2$ *such that for some* $\mathcal{C}_1, \mathcal{C}_2$ *and any* $\gamma > 0$

$$\psi(s, t) = \exp\left\{ \mathbf{i}(\mathcal{C}_1 s + \mathcal{C}_2 t) + \int_{|\mathbf{u}|>\gamma} (e^{\mathbf{i}(su_1+tu_2)} - 1)\nu(du_1, du_2) + \int_{0<|\mathbf{u}|\le\gamma} (e^{\mathbf{i}(su_1+tu_2)} - 1 - \mathbf{i}(su_1 + tu_2))\nu(du_1, du_2) \right\},$$

*where* $\mathbf{u} = (u_1, u_2)$.

This result can be found, for example, in Gikhman and Skorohod [26, Chapter 5]. $\nu$ is called the Lévy measure in the canonical representation of the characteristic function of $\mathbf{Y}$. The stable vectors in our paper will be centered, i.e. $\mathcal{C}_1 = \mathcal{C}_2 = 0$.

**Lemma 3.3.** *If (1.1), (2.1)–(2.4), (2.7) and (3.1) hold, then we have*

$$\lim_{T\to\infty} \frac{T^{\alpha-2}}{L_*(T)} E X_0 I\{|X_0| \le vT\} X_k I\{|X_k| \le wT\}$$
$$= \frac{\alpha}{2-\alpha} \frac{\rho^k}{1-|\rho|^\alpha} (\min(v, w|\rho|^{-k}))^{2-\alpha}.$$

**Proof.** It follows from Theorem 4 of Resnick and Greenwood [43] that

$$\lim_{n\to\infty} nP\left\{\frac{(X_0, X_k)}{b_n} \in \mathbf{A}\right\} = \nu(\mathbf{A}), \tag{3.6}$$

where $b_n$ is defined in (3.2) and $\mathbf{A}$ is any Borel set of $R^2$, not containing (0, 0), $\nu(\mathbf{A}) < \infty$ and the $\nu$-measure of the boundary of $\mathbf{A}$ is 0. Since $nL_*(b_n)/b_n^\alpha \to 1$, with the choice of $n = \lfloor T^\alpha/L_*(T)\rfloor$ we get from (3.6) that

$$\lim_{T\to\infty} \frac{T^\alpha}{L_*(T)} P\{(X_0, X_k)/T \in \mathbf{A}\} = \nu(\mathbf{A}), \tag{3.7}$$

where $\nu$ is the Lévy measure in the canonical representation of the characteristic function of $\mathbf{Z}^{(k)}$. Denoting the joint distribution of $X_0/T$ and $X_k/T$ by $\nu_k^{(T)}$, relation (3.7) says

$$\lim_{T\to\infty} \frac{T^\alpha}{L_*(T)} \mu_k^{(T)}(\mathbf{A}) = \nu(\mathbf{A})$$

for any Borel-set $\mathbf{A} \subset R^2$ not containing (0, 0) and having Lebesgue measure 0 for its boundary. Since the function $f(x, y) = xy$ equals 0 at the origin, using the weak convergence of $T^\alpha \mu_k^{(T)}/L_*(T)$ over $D([-v, v] \times [-w, w] \setminus [-s, s] \times [-s, s])$ and letting $s \downarrow 0$ we get

$$\lim_{T\to\infty} \frac{T^\alpha}{L_*(T)} \int_{-v}^{v}\int_{-w}^{w} xy\, \nu_k^{(T)}(dx, dy) = \int_{-v}^{v}\int_{-w}^{w} xy\, \nu(dx, dy)$$

which can be written equivalently as

$$\lim_{T\to\infty} \frac{T^{\alpha-2}}{L_*(T)} E X_0 I\{|X_0| \le vT\} X_k I\{|X_k| \le wT\} = \int_{-v}^{v}\int_{-w}^{w} xy\, \nu(dx, dy).$$

Since $\xi^{(\alpha)}$ is a stable random variable, its characteristic function can be written as $\exp(-\psi(t))$ and with this notation we get

$$E\exp(\mathbf{i}(sZ_1^{(k)} + tZ_2^{(k)})) = \exp\left(-\sum_{\ell=0}^{\infty} \psi(s\rho^\ell + t\rho^{k+\ell}) - \sum_{\ell=0}^{k-1} \psi(t\rho^\ell)\right).$$

If $\hat{\nu}_\ell$ denotes the Lévy measure associated with the characteristic function $\exp(-\psi(s\rho^\ell + t\rho^{k+\ell}))$ and $\tilde{\nu}_\ell$ corresponds to $\exp(-\psi(t\rho^\ell))$, then we have

$$\nu(\mathbf{A}) = \sum_{\ell=0}^{\infty} \hat{\nu}_\ell(\mathbf{A}) + \sum_{\ell=0}^{k-1} \tilde{\nu}_\ell(\mathbf{A}).$$

Hence

$$\int_{-v}^{v}\int_{-w}^{w} xy\nu(dx,dy) = \sum_{\ell=0}^{\infty}\int_{-v}^{v}\int_{-w}^{w} xy\hat{\nu}_\ell(dx,dy).$$

Next we note that there is a positive constant $A^*$ such that

$$\lim_{x\to\infty}\frac{P\{|\xi^{(\alpha)}|>x\}}{x^{-\alpha}} = A^*$$

and therefore by Bingham et al. [13, p. 346] we obtain that

$$\lim_{x\to\infty}\frac{E(\xi^{(\alpha)})^2 I\{|\xi^{(\alpha)}|\le x\}}{x^2 P\{|\xi^{(\alpha)}|>x\}} = \frac{\alpha}{2-\alpha}$$

resulting in

$$\lim_{x\to\infty}\frac{E(\xi^{(\alpha)})^2 I\{|\xi^{(\alpha)}|\le x\}}{x^{2-\alpha}} = A^*\frac{\alpha}{2-\alpha}.$$

The last relation implies

$$\lim_{T\to\infty} T^{\alpha-2}E\left[\rho^{2\ell+k}(\xi^{(\alpha)})^2 I\{|\xi^{(\alpha)}|\le T\min(v|\rho|^{-\ell}, w|\rho|^{-(\ell+k)})\}\right]$$
$$= A^*\frac{\alpha}{2-\alpha}\rho^{2\ell+k}(\min(v|\rho|^{-\ell}, w|\rho|^{-(\ell+k)}))^{2-\alpha}.$$

We note that $\exp(-\psi(s\rho^\ell + t\rho^{k+\ell}))$ is the characteristic function of the vector $(\rho^\ell\xi^{(\alpha)}, \rho^{k+\ell}\xi^{(\alpha)})$, so repeating the arguments leading to (3.6) and (3.7) for this vector instead of $(X_0, X_k)$ we get

$$\lim_{T\to\infty}\rho^{k+2\ell}\frac{T^{\alpha-2}}{A^*}E\xi^{(\alpha)}I\{|\rho^\ell\xi^{(\alpha)}|\le vT\}\xi^{(\alpha)}I\{|\rho^{\ell+k}\xi^{(\alpha)}|\le wT\}$$
$$= \int_{-v}^{v}\int_{-w}^{w} xy\hat{\nu}_\ell(dx,dy),$$

and therefore

$$\int_{-v}^{v}\int_{-w}^{w} xy\hat{\nu}_\ell(dx,dy) = \frac{\alpha}{2-\alpha}\rho^k|\rho|^{\alpha\ell}(\min(v, w|\rho|^{-k}))^{2-\alpha}.$$

Summing for $\ell = 0, 1, \ldots$, we get Lemma 3.3. □

**Lemma 3.4.** *If (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold, then for every* $k = 0, 1, 2, \ldots$

$$\lim_{n\to\infty}\frac{nE(u_{0,n}(s)-m_n(s))(u_{k,n}(t)-m_n(t))}{A_n^2} = \frac{\alpha}{2-\alpha}\rho^k(\min(s, t|\rho|^{-k}))^{2-\alpha}. \quad (3.8)$$

**Proof.** If $1 < \alpha < 2$, then

$$\lim_{n\to\infty} m_n(t) = EX_0 \quad \text{for any } t > 0. \quad (3.9)$$

If $0 < \alpha < 1$, then

$$|m_n(t)| \le \int_{-tH^{-1}(d/n)}^{tH^{-1}(d/n)} |x|dF(x)$$

$$= -\int_0^{tH^{-1}(d/n)} x dH(x) = -xH(x)\,|_0^{tH^{-1}(d/n)} + \int_0^{tH^{-1}(d/n)} H(x)dx. \tag{3.10}$$

By (3.3) and Bingham et al. [13, p. 26] we have for $0 < \alpha < 1$

$$\lim_{y\to\infty} \frac{\int_0^y H(x)dx}{yH(y)/(1-\alpha)} = 1, \tag{3.11}$$

and therefore

$$m_n(t) = O\left(H^{-1}(d/n)\frac{d}{n}\right). \tag{3.12}$$

If $\alpha = 1$, by assumption $e_0$ is symmetric, so under (1.1) we have that $X_1 = e_1 + c_1$ and therefore

$$\begin{aligned} m_n(t) &= O(1) + E[e_0 I\{|X_0| \le tH^{-1}(d/n)\}] \\ &= O(1) + \int_{tH^{-1}(d/n)-c_1}^{tH^{-1}(d/n)+c_1} x dP\{e_1 \le x\} \\ &= O\left(H^{-1}(d/n)\frac{d}{n}\right) + \int_{tH^{-1}(d/n)-c_1}^{tH^{-1}(d/n)+c_1} P\{e_1 \le x\}dx \\ &= O\left(H^{-1}(d/n)\frac{d}{n}\log H^{-1}(d/n)\right). \end{aligned} \tag{3.13}$$

Thus we get from (3.9)–(3.13) for all $0 < \alpha < 2$ that

$$\frac{nm_n(s)m_n(t)}{A_n^2} \to 0. \tag{3.14}$$

Lemma 3.3 yields

$$\begin{aligned} &\lim_{n\to\infty} \frac{n}{A_n^2}\frac{L(H^{-1}(d/n))}{L_*(H^{-1}(d/n))} EX_0 I\{|X_0| \le sH^{-1}(d/n)\}X_k I\{|X_k| \le tH^{-1}(d/n)\} \\ &= \frac{\alpha}{2-\alpha}\frac{\rho^k}{1-|\rho|^\alpha}(\min(s, t|\rho|^{-k}))^{2-\alpha}. \end{aligned}$$

By (3.4) we have

$$\lim_{n\to\infty} \frac{L(H^{-1}(d/n))}{L_*(H^{-1}(d/n))} = \frac{1}{1-|\rho|^\alpha},$$

which completes the proof of the lemma. □

**Lemma 3.5.** *If* (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) *and* (3.1) *hold, we have for all* $1/2 \le s \le t \le 3/2$ *and* $0 \le x \le 1$ *that*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{A_n^2} E\left(\sum_{k=1}^{\lfloor nx\rfloor}(u_{k,n}(s) - m_n(s))\right)\left(\sum_{k=1}^{\lfloor nx\rfloor}(u_{k,n}(t) - m_n(t))\right) \\ &= x\frac{\alpha}{2-\alpha}\left(s^{2-\alpha} + \sum_{k=1}^{\infty}\rho^k[\min(s, t|\rho|^{-k})^{2-\alpha} + \min(t, s|\rho|^{-k})^{2-\alpha}]\right). \end{aligned}$$

**Proof.** We note that

$$E\left(\sum_{k=1}^{\lfloor nx\rfloor}(u_{k,n}(s)-m_n(s))\right)\left(\sum_{k=1}^{\lfloor nx\rfloor}(u_{k,n}(t)-m_n(t))\right)$$
$$=\lfloor nx\rfloor E(u_{0,n}(s)-m_n(s))(u_{0,n}(t)-m_n(t))$$
$$+\sum_{k=1}^{\lfloor nx\rfloor-1}(\lfloor nx\rfloor-k)E(u_{0,n}(s)-m_n(s))(u_{k,n}(t)-m_n(t))$$
$$+\sum_{k=1}^{\lfloor nx\rfloor-1}(\lfloor nx\rfloor-k)E(u_{0,n}(t)-m_n(t))(u_{k,n}(s)-m_n(s)).$$

Let

$$e_k^*=\sum_{\ell=0}^{k-1}\rho^\ell\varepsilon_{k-\ell}\quad\text{and}\quad X_k^*=c_1+e_k^*. \tag{3.15}$$

Using again Theorem 2.3 of Cline [16] (cf. also [22]), it follows that there is a constant $C_1$ such that

$$P\{|X_k^*|>x\}\le C_1x^{-\alpha}L(x)\quad\text{for all } k \text{ and } 0\le x<\infty. \tag{3.16}$$

Clearly as in (3.5),

$$X_k-X_k^*=e_k-e_k^*=\sum_{\ell=k}^{\infty}\rho^\ell\varepsilon_{k-\ell}=\sum_{j=0}^{\infty}\rho^{k+j}\varepsilon_{-j}=\rho^k(X_0-c_1). \tag{3.17}$$

Next we write

$$|E(u_{0,n}(s)-m_n(s))(u_{k,n}(t)-m_n(t))|=|Eu_{0,n}(t)u_{0,n}(s)-m_n(t)m_n(s)|$$
$$\le|E(X_{0,n}(X_k-X_k^*)I\{|X_0|\le sH^{-1}(d/n)\}I\{|X_k|\le tH^{-1}(d/n)\})|$$
$$+|E(X_0X_k^*I\{|X_0|\le sH^{-1}(d/n)\}I\{|X_k|\le tH^{-1}(d/n)\})-m_n(s)m_n(t)|$$
$$\le A_{1,k,n}+A_{2,k,n}+A_{3,k,n}$$

with

$$A_{1,k,n}=E|X_0(X_k-X_k^*)I\{|X_0|\le sH^{-1}(d/n)\}I\{|X_k|\le tH^{-1}(d/n)\}|,$$
$$A_{2,k,n}=E\big[|X_0X_k^*|I\{|X_0|\le sH^{-1}(d/n)\}|I\{|X_k|\le tH^{-1}(d/n)\}-I\{|X_k^*|\le tH^{-1}(d/n)\}|\big]$$

and

$$A_{3,k,n}=|E(X_0X_k^*I\{|X_0|\le sH^{-1}(d/n)\}I\{|X_k^*|\le tH^{-1}(d/n)\})-m_n(s)m_n(t)|.$$

Using (3.14) and (3.17) we conclude

$$A_{1,k,n}\le|\rho|^kE|X_0|\,|X_0-c_1|I\{|X_0|\le sH^{-1}(d/n)\}$$
$$\le C_2|\rho|^k(H^{-1}(d/n))^2d/n \tag{3.18}$$

with some constant $C_2$. Next we note that

$$A_{2,k,n}\le E\big[|X_0X_k^*|I\{|X_0|\le sH^{-1}(d/n)\}I\{tH^{-1}(d/n)-|\rho|^k|X_0|\le|X_k^*|\le tH^{-1}(d/n)\}\big]$$

$$+ E\big[|X_0 X_k^*| I\{|X_0| \le s H^{-1}(d/n)\} I\{t H^{-1}(d/n) \le |X_k^*| \le t H^{-1}(d/n) + |\rho|^k |X_0|\}\big]$$
$$= A_{2,k,n}^{(1)} + A_{2,k,n}^{(2)}. \tag{3.19}$$

Using the independence of $X_0$ and $X_k^*$ we get

$$A_{2,k,n}^{(1)} \le E|X_0| I\{|X_0| \le s H^{-1}(d/n)\} E|X_k^*| I\{t H^{-1}(d/n) - |\rho|^k H^{-1}(d/n) \le |X_k^*| \le t H^{-1}(d/n)\}.$$

By (3.16) we have that

$$\begin{aligned} &E|X_k^*| I\{t H^{-1}(d/n) - |\rho|^k H^{-1}(d/n) \le |X_k^*| \le t H^{-1}(d/n)\} \\ &= -x P\{|X_k^*| > x\}\Big|_{tH^{-1}(d/n)-|\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)} + \int_{tH^{-1}(d/n)-|\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)} P\{|X_k^*| > x\} dx \\ &\le \int_{tH^{-1}(d/n)-|\rho|^k H^{-1}(d/n)}^{tH^{-1}(d/n)} P\{|X_k^*| > x\} dx \\ &\le C_3 |\rho|^k H^{-1}(d/n) d/n, \end{aligned} \tag{3.20}$$

where $C_3$ is a constant. Hence, on account of (3.9), (3.12) and (3.13) we obtain that with some constant $C_4$

$$A_{2,k,n}^{(1)} \le C_4 \rho^k (H^{-1}(d/n))^2 d/n$$

and similarly

$$A_{2,k,n}^{(2)} \le C_4 \rho^k (H^{-1}(d/n))^2 d/n,$$

resulting in

$$A_{2,k,n} \le C_5 \rho^k (H^{-1}(d/n))^2 d/n. \tag{3.21}$$

Using again the independence of $X_0$ and $X_k^*$ we get

$$A_{3,k,n} = |m_n(s)|\, |E X_k^* I\{|X_k^*| \le t H^{-1}(d/n)\} - m_n(t)|.$$

It is easy to see that

$$\begin{aligned} E X_k^* I\{|X_k^*| \le t H^{-1}(d/n)\} &= E X_k^* I\{|X_k^*| \le t H^{-1}(d/n)\} I\{|X_0| > |\rho|^{-k/2} H^{-1}(d/n)\} \\ &\quad + E X_k^* I\{|X_k^*| \le t H^{-1}(d/n)\} I\{|X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\} \end{aligned}$$

and by the independence of $X_0$ and $X_k^*$ and (3.16) we have

$$\begin{aligned} &|E X_k^* I\{|X_k^*| \le t H^{-1}(d/n)\} I\{|X_0| > |\rho|^{-k/2} H^{-1}(d/n)\}| \\ &\le C_5 |m_n(t)| H(|\rho|^{-k/2} H^{-1}(d/n)) \\ &\le C_6 |m_n(t)|\, |\rho|^{k\alpha/2} d/n. \end{aligned}$$

Next we note that

$$\begin{aligned} \Big| E\big[X_k^* I\{|X_k^*| \le t H^{-1}(d/n), |X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\}\big] \\ - E\big[(X_k^* + \rho^k (X_0 - c_1)) I\big\{|X_k^* + \rho^k (X_0 - c_1)| \le t H^{-1}(d/n), \\ |X_0| \le |\rho|^{-k/2} H^{-1}(d/n)\big\}\big]\Big| \end{aligned}$$

$$\leq |\rho|^k E\big[|X_0 - c_1| I\{|X_k^* + \rho^k (X_0 - c_1)| \leq t H^{-1}(d/n),$$
$$|X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}\big]$$
$$+ E\big[|X_k^*|\,|I\{|X_k^*| \leq t H^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}$$
$$- I\{|X_k^* + \rho^k (X_0 - c_1)| \leq t H^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}|\big]$$
$$\leq |\rho|^k (|\rho|^{-k/2} H^{-1}(d/n) + |c_1|)$$
$$+ E|X_k^*| I\{(t - |\rho|^{k/2}) H^{-1}(d/n) - |c_1|\,|\rho|^k \leq |X_k^*| \leq t H^{-1}(d/n)\}$$
$$+ E|X_k^*| I\{t H^{-1}(d/n) \leq |X_k^*| \leq (t + |\rho|^{-k/2}) H^{-1}(d/n) + |c_1|\,|\rho|^k\}$$
$$\leq C_7(|\rho|^{k/2} H^{-1}(d/n) + |\rho|^k H^{-1}(d/n) d/n)$$

by (3.20). Similarly

$$|E X_k I\{|X_k| \leq t H^{-1}(d/n)\} - E X_k I\{|X_k| \leq t H^{-1}(d/n), |X_0| \leq |\rho|^{-k/2} H^{-1}(d/n)\}|$$
$$\leq C_8(|\rho|^{k/2} H^{-1}(d/n) + |\rho|^k H^{-1}(d/n) d/n).$$

Hence

$$A_{3,k,n} \leq C_9 |\rho|^{\tau k} (H^{-1}(d/n))^2 d/n, \quad \text{where } \tau = \min\{1, \alpha\}/2. \tag{3.22}$$

Putting together (3.18), (3.21) and (3.22) we get that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \frac{1}{A_n^2} \sum_{k=K}^{\lfloor nx \rfloor - 1} |(\lfloor nx \rfloor - k) E(u_{0,n}(s) - m_n(s))(u_{k,n}(t) - m_n(t))| = 0. \tag{3.23}$$

The lemma now follows from Lemma 3.4 and (3.23). □

## 4. A weak convergence result

Define the two-parameter process

$$L_n(t, x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx \rfloor} (X_i I\{|X_i| \leq t H^{-1}(d/n)\} - m_n(t)),$$

for $0 \leq x \leq 1$, $1/2 \leq t \leq 3/2$. First we show the tightness of $L_n(t, x)$. The proof is based on a generalization of [10]. We introduce

$$X_{i,1} = \max(X_i, 0), \qquad X_{i,2} = \min(X_i, 0)$$

and

$$m_{n,1}(t) = E X_{0,1} I\{|X_0| \leq t H^{-1}(d/n)\}, \qquad m_{n,2}(t) = E X_{0,2} I\{|X_0| \leq t H^{-1}(d/n)\}.$$

Similarly to $L_n(t, x)$, we define

$$L_{n,1}(t, x) = \frac{1}{A_n} \sum_{i=1}^{\lfloor nx \rfloor} (X_{i,1} I\{|X_i| \leq t H^{-1}(d/n)\} - m_{n,1}(t)),$$

and $L_{n,2}(t, x)$ is defined in a similar fashion. Clearly, if both $L_{n,1}$ and $L_{n,2}$ are tight, then $L_n(t, x)$ is tight as well. We prove only tightness of $L_{n,1}$, the same argument can be used in case of $L_{n,2}$. Let

$$g_n = \frac{1}{d^{1/2} \log\log n}.$$

**Lemma 4.1.** *If (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold, then*

$$m_{n,1}(t) \text{ is a non-decreasing function on } [1/2, 3/2], \tag{4.1}$$

$$\frac{n}{A_n} \sup_{|t_2 - t_1| \le g_n} |m_{n,1}(t_2) - m_{n,1}(t_1)| \to 0, \quad n \to \infty, \tag{4.2}$$

$$E|L_{n,1}(t_2, x) - L_{n,1}(t_1, x)|^6 \le C_1 |t_2 - t_1|^\tau, \quad \text{if } |t_2 - t_1| \ge g_n, \tag{4.3}$$

*and*

$$E|L_{n,1}(t, x_2) - L_{n,1}(t, x_1)|^6 \le C_1 |x_2 - x_1|^\tau, \quad \text{if } |x_2 - x_1| \ge g_n, \tag{4.4}$$

*with some $\tau > 2$ and constant $C_1$.*

**Proof.** The definition of $m_{n,1}(t)$ implies immediately (4.1).

By the definition of $m_{n,1}(t)$ we have for all $1/2 \le t_1 \le t_2 \le 3/2$ that

$$\begin{aligned}
0 \le m_{n,1}(t_2) - m_{n,1}(t_1) &= EX_{0,1}(I\{t_1 H^{-1}(d/n) < |X_0| \le t_2 H^{-1}(d/n)\}) \\
&\le \int_{t_1 H^{-1}(d/n)}^{t_2 H^{-1}(d/n)} x dH(x) \\
&\le C_2 \big( |t_2 H^{-1}(d/n) H(t_2 H^{-1}(d/n)) - t_1 H^{-1}(d/n) H(t_1 H^{-1}(d/n))| \\
&\quad + |t_2 - t_1| H^{-1}(d/n) H(t_1 H^{-1}(d/n)) \big) \\
&\le C_3 |t_2 - t_1| \frac{d}{n} H^{-1}(d/n)
\end{aligned}$$

on account of integration by parts and (3.3), establishing (4.2).

Next we introduce

$$Y_i = \sum_{k=0}^{\lfloor K \log n \rfloor} \rho^k \varepsilon_{i-k} + c_1, \qquad Y_{i,1} = \max(Y_i, 0) \tag{4.5}$$

and $\xi_i = \eta_i - E\eta_i$ with

$$\eta_i = \eta_i(t_1, t_2) = Y_{i,1} I\{t_1 H^{-1}(d/n) < |Y_i| \le t_2 H^{-1}(d/n)\}.$$

Since $E|\varepsilon_0|^{\alpha/2} < \infty$, using Markov's inequality we see that for every $\beta > 0$ there is a constant $K = K(\beta)$ such that

$$\left| E(L_{n,1}(t_2, x) - L_{n,1}(t_1, x))^6 - \frac{1}{A_n^6} \sum_{1 \le i_1, \ldots, i_6 \le \lfloor nx \rfloor} E\xi_{i_1} \ldots \xi_{i_6} \right| \le C_5 n^{-\beta}. \tag{4.6}$$

We note that by definition, $\{\xi_i\}$ is a stationary, $\lfloor K \log n \rfloor$-dependent sequence with zero mean. Let us divide the indices $i_1, \ldots, i_6$ into groups so that the difference between the indices within a group is less than $\lfloor K \log n \rfloor$ and between groups is larger than $\lfloor K \log n \rfloor$. Clearly $E\xi_{i_1} \ldots \xi_{i_6} = 0$, if there is at least one group containing a single element. So it suffices to consider the cases when all groups contain at least two elements. This allows the cases of one single group with 6 elements ($D_1$), two groups with $3 + 3$ ($D_2$) or $4 + 2$ ($D_3$) elements and finally 3 groups with 2 elements in each ($D_4$). If there is only one group, then via Hölder's inequality we have

$$|E\xi_{i_1} \ldots \xi_{i_6}| \le E|\xi_0|^6 \le 2^6 (E|\eta_0|^6 + |E\eta_0|^6).$$

Since the cardinality of $D_1$ is bounded by constant times $n(\log n)^5$ we conclude

$$\left|\frac{1}{A_n^6}\sum_{D_1} E\xi_{i_1}\ldots\xi_{i_6}\right| \le C_6\bigg(\frac{n(\log n)^5}{A_n^6}[EX_0^6 I\{t_1 H^{-1}(d/n)\le |X_0|\le t_2 H^{-1}(d/n)\} + (EX_0 I\{t_1 H^{-1}(d/n)\le |X_0|\le t_2 H^{-1}(d/n)\})^6] + n^{-\beta}\bigg).$$

Integration by parts and (3.3) yield

$$EX_0^6 I\{t_1 H^{-1}(d/n)\le |X_0|\le t_2 H^{-1}(d/n)\} \le C_7|t_2-t_1|\frac{d}{n}(H^{-1}(d/n))^6,$$

resulting in

$$\left|\frac{1}{A_n^6}\sum_{D_1} E\xi_{i_1}\ldots\xi_{i_6}\right| \le C_8\left(\frac{(\log n)^5}{d^2}|t_2-t_1| + n^{-\beta}\right).$$

Using again the $\lfloor K\log n\rfloor$ dependence of $\{\xi_i\}$ and the fact that the cardinality of $D_2$ is constant times $n^2(\log n)^4$ we conclude via Hölder's inequality

$$\begin{aligned}\left|\frac{1}{A_n^6}\sum_{D_2} E\xi_{i_1}\ldots\xi_{i_6}\right| &= \left|\frac{1}{A_n^6}\sum_{D_2} E\xi_{i_1}\xi_{i_2}\xi_{i_3}E\xi_{i_4}\xi_{i_5}\xi_{i_6}\right| \\ &\le C_8\bigg(\frac{n^2(\log n)^4}{A_n^6}[EX_0^3 I\{t_1 H^{-1}(d/n)\le |X_0|\le t_2 H^{-1}(d/n)\} \\ &\quad + (EX_0 I\{t_1 H^{-1}(d/n)\le |X_0|\le t_2 H^{-1}(d/n)\})^3]^2 + n^{-\beta}\bigg) \\ &\le C_9\left(\frac{(\log n)^4}{d}(t_2-t_1)^2 + n^{-\beta}\right).\end{aligned}$$

Similar arguments give

$$\left|\frac{1}{A_n^6}\sum_{D_3} E\xi_{i_1}\ldots\xi_{i_6}\right| \le C_{10}\left(\frac{(\log n)^4}{d}(t_2-t_1)^2 + n^{-\beta}\right).$$

Following the proof of Lemma 3.5 we obtain

$$\begin{aligned}\left|\frac{1}{A_n^6}\sum_{D_4} E\xi_{i_1}\ldots\xi_{i_6}\right| &\le C_{11}\left(\frac{1}{A_n^6}\left(n\sum_{i=0}^{\infty}\xi_0\xi_i\right)^3 + n^{-\beta}\right) \\ &\le C_{11}\left(|t_2-t_1|^3 + n^{-\beta}\right).\end{aligned}$$

Putting together our estimates and using the choice of $g_n$ we conclude for all $|t_2-t_1|\ge g_n$

$$\begin{aligned}E(L_{n,1}(t_2,x)-L_{n,1}(t_1,x))^6 &\le C_{12}\bigg(\frac{(\log n)^5}{d^2}|t_2-t_1| + \frac{(\log n)^4}{d}|t_2-t_1|^2 \\ &\quad + |t_2-t_1|^3 + n^{-\beta}\bigg) \\ &\le C_{13}|t_2-t_1|^{\tau}\end{aligned}$$

with any $2<\tau\le 3$ on account of assumption (2.10). Hence the proof of (4.3) is complete.

The proof of (4.4) goes along the lines of the arguments used to establish (4.3) and therefore it is omitted. □

**Lemma 4.2.** *If (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold, then* $L_n(t, x)$ *is tight in* $\mathcal{D}([1/2, 3/2] \times [0, 1])$.

**Proof.** It follows from a minor modification of Lemma 6 in [10] that both $L_{n,1}$ and $L_{n,2}$ are tight. Since $L_n = L_{n,1} + L_{n,2}$, the result is proven. □

Next we consider the convergence of the finite dimensional distributions. It is based in the following lemma:

**Lemma 4.3.** *We assume that (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) and (3.1) hold. Let* $N = \lfloor (\log n)^\gamma \rfloor$ *with some* $\gamma > 0$. *Then*

$$E\left(\sum_{i=1}^{N}(X_i I\{|X_i| \le tH^{-1}(d/n)\} - E[X_i I\{|X_i| \le tH^{-1}(d/n)\}])\right)^4$$

$$\le C_{13}\left(N(\log N)^3 (H^{-1}(d/n))^4 \frac{d}{n} + N^2 (H^{-1}(d/n))^4 \left(\frac{d}{n}\right)^2\right) \tag{4.7}$$

*with some constant* $C_{13}$ *and*

$$\lim_{n\to\infty} \frac{Nn}{A_n^2} E\left(\sum_{k=1}^{N}(u_{k,n}(s) - m_n(s))\right)\left(\sum_{k=1}^{N}(u_{k,n}(t) - m_n(t))\right)$$

$$= \frac{\alpha}{2-\alpha}\left(s^{2-\alpha} + \sum_{k=1}^{\infty}\rho^k[\min(s, t|\rho|^{-k})^{2-\alpha} + \min(t, s|\rho|^{-k})^{2-\alpha}]\right). \tag{4.8}$$

**Proof.** We recall the definition of $\xi_i$ from the proof of Lemma 4.1. For any $\beta > 0$, choosing $K$ in the definition of $Y_i$ in (4.5) we get that

$$E\left(\sum_{i=1}^{N}(X_i I\{|X_i| \le tH^{-1}(d/n)\} - E[X_i I\{|X_i| \le tH^{-1}(d/n)\}])\right)^4$$

$$\le C_{14}\left(E\left(\sum_{i=1}^{N}\xi_i\right)^4 + n^{-\beta}\right).$$

We write

$$E\left(\sum_{i=1}^{N}\xi_i\right)^4 = \sum_{i_1,\ldots,i_4}^{N} E\xi_{i_1}\ldots\xi_{i_4}.$$

We note again that $\{\xi_i\}$ is a stationary $K\log n$ dependent sequence with 0 mean. Let us divide the indices $i_1, \ldots, i_4$ into blocks so that the difference between the indices within a block is less than $K\log n$ and between blocks is larger than $K\log n$. Clearly $E\xi_{i_1}\ldots\xi_{i_4} = 0$, if there is at least one block containing only a single element. So we need to consider the cases of one single block with 4 elements ($D_1$) and two blocks with $2+2$ elements ($D_2$). The number of the elements in $D_1$ is not greater than constant times $N(\log N)^3$ and as we showed in the proof of Lemma 4.1

$$E\xi_0^4 \le C_{14}\left((H^{-1}(d/n))^4\frac{d}{n} + n^{-\beta}\right),$$

assuming that $K$ in (4.5) is sufficiently large. Hence

$$\left|\sum_{D_1}^{N} E\xi_{i_1}\ldots\xi_{i_4}\right| \le C_{15}\left(N(\log N)^3(H^{-1}(d/n))^4\frac{d}{n}+n^{-\beta}\right).$$

As in the proof of Lemma 4.1 we get that

$$\left|\sum_{D_2}^{N} E\xi_{i_1}\ldots\xi_{i_4}\right| \le C_{16}N^2\left(\sum_{i=0}^{\infty}|E\xi_0\xi_i|\right)^2$$

and

$$\sum_{i=0}^{\infty}|E\xi_0\xi_i| \le \left(C_{17}(H^{-1}(d/n))^2\frac{d}{n}+n^{-\beta}\right),$$

completing the proof of (4.7). The proof of (4.8) goes along the lines of the arguments used to establish Lemma 3.5. □

**Lemma 4.4.** *If* (1.1), (2.1)–(2.4), (2.7), (2.9), (2.10) *and* (3.1) *hold, then*

$$L_n(t,x) \longrightarrow \Gamma(t,x) \quad \text{weakly in } \mathcal{D}([1/2,3/2]\times[0,1]),$$

*where* $\Gamma(t,x)$ *is a Gaussian process with* $E\Gamma(t,x)=0$ *and*

$$E\Gamma(t,x)\Gamma(s,y)$$
$$=\min(x,y)\frac{\alpha}{2-\alpha}\left((\min(s,t))^{2-\alpha}+\sum_{k=1}^{\infty}\rho^k[\min(s,t|\rho|^{-k})^{2-\alpha}+\min(t,s|\rho|^{-k})^{2-\alpha}]\right).$$

**Proof.** By Lemma 4.2, the process $L_n(t,x)$ is tight, so we need only to show the convergence of the finite dimensional distributions. By the Cramér–Wold device it is sufficient to prove the asymptotic normality of

$$Q_n=\sum_{j=1}^{J}\sum_{\ell=0}^{L}\mu_{j,\ell}(L_n(t_j,x_{\ell+1})-L_n(t_j,x_\ell))$$

for all $J$, $L$, real coefficients $\mu_{j,\ell}$, $1/2\le t_j\le 3/2$, $1\le j\le J$, and $0=x_0<x_1<\cdots<x_L<x_{L+1}=1$. We recall the definition of $X_k^*$ from the proof of Lemma 3.5 (cf. (3.17)) and define

$$\bar{L}_n(t,x)=\frac{1}{A_n}\sum_{i=1}^{\lfloor nx\rfloor}(X_k^*I\{|X_k^*|\le tH^{-1}(d/n)\}-EX_k^*I\{|X_k^*|\le tH^{-1}(d/n)\}).$$

Choosing $K$ large enough in the definition of $X_k^*$, we get from the arguments used in the proof of Lemmas 3.5, 4.1 and 4.3 that

$$E(L_n(t,x)-\bar{L}_n(t,x))^2\to 0.$$

So we need to establish only the asymptotic normality of

$$\bar{Q}_n=\sum_{\ell=0}^{L}\sum_{j=1}^{J}\mu_{j,\ell}(\bar{L}_n(t_j,x_{\ell+1})-\bar{L}_n(t_j,x_\ell)).$$

Let

$$z_{k,\ell} = \sum_{j=1}^{J} \mu_{j,\ell}(X_k^* I\{|X_k^*| \le t_j H^{-1}(d/n)\} - E[X_k^* I\{|X_k^*| \le t_j H^{-1}(d/n)\}]).$$

Since for all $\ell$

$$E\left(\frac{1}{A_n}\sum_{k=1}^{\lfloor K \log n\rfloor} z_{k,\ell}\right)^2 \to 0,$$

by stationarity and the $\lfloor K \log n\rfloor$-dependence of $z_{k,\ell}$ for any $\ell$ we get that the variables

$$\frac{1}{A_n}\sum_{k=\lfloor nx_\ell\rfloor+1}^{\lfloor nx_{\ell+1}\rfloor} z_{k,\ell}, \quad 1 \le \ell \le L \text{ are asymptotically independent.}$$

By stationarity we have

$$\frac{1}{A_n}\sum_{k=\lfloor nx_\ell\rfloor+1}^{\lfloor nx_{\ell+1}\rfloor} z_{k,\ell} \stackrel{\mathcal{D}}{=} \frac{1}{A_n}\sum_{k=1}^{\lfloor nx_{\ell+1}\rfloor-\lfloor nx_\ell\rfloor} z_{k,\ell}.$$

Let us divide the integers of $[1, \lfloor nx_{\ell+1}\rfloor - \lfloor nx_\ell\rfloor]$ into consecutive blocks $R_1, V_1, R_2, V_2, \dots, R_s, V_s$ such that for $1 \le i \le s-1$, $R_i$ contains $\lfloor(\log n)^\gamma\rfloor$ integers, $V_i$ contains $\lfloor K \log n\rfloor$ integers, the last two blocks might contain less elements. Let

$$\zeta_{i,1} = \sum_{k\in R_i} z_{k,\ell} \quad \text{and} \quad \zeta_{i,2} = \sum_{k\in V_i} z_{k,\ell}.$$

Due to the $\lfloor K \log n\rfloor$ dependence and stationarity, the variables $\zeta_{i,2}$, $1 \le i < s$ are independent and identically distributed and the proof of Lemma 3.5 shows that

$$E\left(\frac{1}{A_n}\sum_{i=1}^{s} \zeta_{i,2}\right)^2 \to 0.$$

Using Lemma 4.3 we get that

$$E\zeta_{i,1}^2 \ge C_{18}(\log n)^\gamma (H^{-1}(d/n))^2 d/n$$

and

$$E\zeta_{i,1}^2 \le C_{19}\left((\log n)^\gamma (\log\log n)^3 (H^{-1}(d/n))^4 \frac{d}{n} + (\log n)^{2\gamma}(H^{-1}(d/n))^4\left(\frac{d}{n}\right)^2\right).$$

Since $s$ is proportional to $n/(\log n)^\gamma$, a simple calculation yields

$$\frac{\sum_{i=1}^{s} E\zeta_{i,1}^4}{\left(\sum_{i=1}^{s} E\zeta_{i,1}^2\right)^2} \to 0.$$

Thus the central limit theorem with Lyapunov's remainder term (cf. [41, p. 154]) implies the asymptotic normality of $\sum_{1\le k\le\lfloor nx_{\ell+1}\rfloor-\lfloor nx_\ell\rfloor} z_{k,\ell}$. This completes the proof of Lemma 4.4. □

## 5. Proof of Theorems 2.1 and 2.2

We need the weak law of large numbers for $\eta_{d,n}$.

**Lemma 5.1.** *If (1.1), (2.1)–(2.4), (2.7), (2.9)–(2.11) and (3.1) hold, then we have*

$$\frac{\eta_{d,n}}{H^{-1}(d/n)} \xrightarrow{P} 1.$$

**Proof.** Using Gorodetskii [27] and Withers [46] we get that $X_k$ is a strongly mixing stationary sequence with mixing rate $\alpha(k) \le C_1 \exp(-\lambda k)$ for some $C_1 > 0$ and $\lambda > 0$. Fix $1/2 < t < 2$ and let $T_k = I\{|X_k| \ge tH^{-1}(d/n)\}, 1 \le k \le n$. Clearly, $ET_k = P\{|X_k| \ge tH^{-1}(d/n)\} = H(tH^{-1}(d/n))$ and due to the regular variation of $H$, $ET_k/(d/n) \to t^{-\alpha}$, as $n \to \infty$. On the other hand, by the correlation inequality of Davydov [23] we get for any $p > 2$ that

$$\begin{aligned}|ET_0T_k - ET_0ET_k| &\le (\alpha(k))^{(p-1)/p}(ET_0^p)^{1/p}(ET_k^p)^{1/p}\\ &\le C_1 \exp(-\lambda k(p-1)/p)(ET_0^p)^{2/p}\\ &= C_1 \exp(-\lambda k(p-1)/p)(ET_0)^{2/p}\\ &\le C_2 \exp(-\lambda k(p-1)/p)(d/n)^{2/p}.\end{aligned}$$

Hence setting $\bar{T}_k = T_k - ET_k$ we conclude that

$$\begin{aligned}E\left(\sum_{k=1}^{n}\bar{T}_k\right)^2 &= nE\bar{T}_0^2 + 2\sum_{k=1}^{n-1}(n-k)E\bar{T}_0\bar{T}_k\\ &\le n\left(E\bar{T}_k^2 + 2\sum_{k=1}^{n-1}|E\bar{T}_0\bar{T}_k|\right)\\ &\le n\left(ET_0^2 + C_3\sum_{k=1}^{n-1}\exp(-\lambda k(p-1)/p)(d/n)^{2/p}\right)\\ &\le n\left(ET_0 + C_5(d/n)^{2/p}\right)\\ &\le n(d/n)^{2/p}.\end{aligned}$$

Thus by Markov's inequality we have that

$$P\left\{\sum_{k=1}^{n}\bar{T}_k \ge d^{2/p}\right\} \le C_6 n^{(p-2)/p}/d^{2/p} \to 0,$$

provided that $d/n^{(2-p)/p} \to 0$. Since $d \ge n^{\delta}$, choosing $p$ near 2, it follows that

$$\sum_{k=1}^{n} T_k = t^{-\alpha}d(1+o_P(1)) + o_P(d^{2/p}) = t^{-\alpha}d(1+o_P(1)).$$

In other words,

$$\frac{1}{d}\#\{k \le n : |X_k| \ge tH^{-1}(d/n)\} \xrightarrow{P} t^{-\alpha}, \quad \text{as } n \to \infty.$$

This shows that

$$\lim_{n\to\infty} P\{\eta_{n,d} \ge tH^{-1}(d/n)\} = 1 \quad \text{for } t < 1$$

and

$$\lim_{n\to\infty} P\{\eta_{n,d} \geq tH^{-1}(d/n)\} = 0 \quad \text{for } t > 1,$$

completing the proof of Lemma 5.1. □

**Proof of Theorem 2.1.** We note that $\Gamma(t, x)$ is a continuous process. Hence combining Lemmas 4.4 and 5.1 we conclude

$$L_n(\eta_{d,n}/H^{-1}(d/n), x) \stackrel{\mathcal{D}[0,1]}{\longrightarrow} \Gamma(1, x).$$

It is easy to see that

$$\{\Gamma(1, x), 0 \leq x \leq 1\} \stackrel{\mathcal{D}}{=} \left\{\left(\frac{\alpha}{2-\alpha}\frac{1+\rho}{1-\rho}\right)^{1/2} W(x),\ 0 \leq x \leq 1\right\},$$

where $W(x)$ is a Wiener process, which completes the proof. □

**Proof of Theorem 2.2.** Since

$$\frac{1}{A_n}T_{n,d}(x) = L_n(\eta_{d,n}/H^{-1}(d/n), x) - \frac{\lfloor nx\rfloor}{n}L_n(\eta_{d,n}/H^{-1}(d/n), 1),$$

Theorem 2.1 yields

$$\frac{1}{A_n}T_{n,d}(x) \stackrel{\mathcal{D}[0,1]}{\longrightarrow} \left(\frac{\alpha}{2-\alpha}\frac{1+\rho}{1-\rho}\right)^{1/2}(W(x) - xW(1)).$$

By definition, $B(x) = W(x) - xW(1), 0 \leq x \leq 1$ is a Brownian bridge, so the proof of Theorem 2.2 is complete. □

## Acknowledgments

We would like to thank the AE and two referees for their comments leading to a substantial improvement of the presentation. This research was supported by Austrian Science Fund grants P24302-N18, W1230, Hungarian Scientific Research Fund grant K 108615 and National Science Foundation grant DMS 1305858.

## References

[1] B. Andrews, M. Calder, R.A. Davis, Maximum likelihood estimation for $\alpha$-stable autoregressive processes, Ann. Statist. 37 (2009) 1946–1982.

[2] D.Z. Arov, A,A. Bobrov, The extreme terms of a sample and their role in the sum of independent variables, Theory Probab. Appl. 5 (1960) 377–396.

[3] A. Aue, I. Berkes, L. Horváth, Selection from a stable box, Bernoulli 14 (2008) 125–139.

[4] A. Aue, L. Horváth, Stuctural breaks in time series, J. Time Ser. Anal. 34 (2013) 1–16.

[5] A. Aue, L. Horváth, A limit theorem for mildly explosive autoregression with stable errors, Econometric Theory 23 (2007) 201–220.

[6] F.S. Avram, M.S. Taqqu, Weak convergence of moving averages with infinite variance, in: E. Eberlein, M.S. Taqqu (Eds.), Dependence in Probability and Statistics: A Survey of Recent Results, Birkhäuser, Boston, 1986, pp. 399–425.

[7] F.S. Avram, M.S. Taqqu, Weak convergence of moving averages in the $\alpha$-stable domain of attraction, Ann. Probab. 20 (1992) 483–503.

[8] B. Basrak, D. Krizmanič, J. Segers, A functional limit theorem for partial sums of dependent random variables with infinite variance, Ann. Probab. 40 (2012) 2008–2033.

[9] I. Berkes, L. Horváth, The central limit theorem for sums of trimmed variables with heavy tails, Stochastic Process. Appl. 122 (2012) 449–465.
[10] I. Berkes, L. Horváth, S. Ling, J. Schauer, Testing for structural change of AR model to threshold AR model, J. Time Ser. Anal. 32 (2011) 547–565.
[11] I. Berkes, L. Horváth, J. Schauer, Asymptotics of trimmed CUSUM statistics, Bernoulli 17 (2011) 1344–1367.
[12] P. Billingsley, Convergence of Probability Measures, Wiley, 1968.
[13] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University Press, 1987.
[14] N.H. Chan, Inference for near-integrated time series with infinite variance, J. Amer. Statist. Assoc. 85 (1990) 1069–1074.
[15] N.H. Chan, L.T. Tran, On the first-order autoregressive processes with infinite variance, Econometric Theory 5 (1989) 354–362.
[16] D. Cline, Infinite Series of Random Variables With Regularly Varying Tails, Techical Report 83-24, The Institute of Applied Mathematics and Statistics, The University of British Columbia, Vancouver, Canada, 1983.
[17] M. Csörgő, L. Horváth, Limit Theorems in Change-Point Analysis, Wiley, 1998.
[18] S. Csörgő, L. Horváth, D.M. Mason, What portion of the sample makes a partial sum asymptotically stable or normal? Probab. Theory Related Fields 72 (1986) 1–16.
[19] D.A. Darling, The influence of the maximum term in the addition of independent random variables, Trans. Amer. Math. Soc. 73 (1952) 95–107.
[20] R.A. Davis, T. Mikosch, The sample autocorrelations of heavy-tailed processes with applications to ARCH, Ann. Statist. 26 (1998) 2049–2080.
[21] R. Davis, S. Resnick, Limit theory for moving averages of random variables with regularly varying tail probabilities, Ann. Probab. 13 (1985) 179–195.
[22] R. Davis, S. Resnick, Limit theory for the sample covariance and correlation functions of moving averages, Ann. Statist. 14 (1986) 533–558.
[23] Y.A. Davydov, Convergence of distributions generated by stationary stochastic processes, Theory Probab. Appl. 13 (1968) 691–696.
[24] A. de Acosta, E. Giné, Convergence of moments and related functionals in the general central limit theorem in Banach spaces, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 48 (1979) 213–231.
[25] E. Fama, The behavior of stock market price, J. Bus. 38 (1965) 34–105.
[26] I. Gikhman, A.V. Skorohod, Introduction to the Theory of Random Processes, Saunders, Philadelphia.
[27] V.V. Gorodetskii, On the strong mixing property for linear sequences, Theory Probab. Appl. 22 (1977) 411–413.
[28] P. Griffin, W.E. Pruitt, The central limit problem for trimmed sums, Math. Proc. Cambridge Philos. Soc. 102 (1987) 329–349.
[29] P. Griffin, W.E. Pruitt, Asymptotic normality and subsequential limits of trimmed sums, Ann. Probab. 17 (1989) 1186–1219.
[30] P. Hall, On the extreme terms of a sample from the domain of attraction of a stable law, J. Lond. Math. Soc. 18 (1978) 181–191.
[31] L. Horváth, R. Reeder, Detecting changes in functional linear models, J. Multivariate Anal. 111 (2012) 310–334.
[32] H. Kesten, Convergence in distribution of lightly trimmed and untrimmed sums are equivalent, Math. Proc. Cambridge Philos. Soc. 113 (1993) 615–638.
[33] R. Kulik, M. Ould Haye, Trimmed sums of long range dependent moving averages, Statist. Probab. Lett. 78 (2008) 2536–2542.
[34] R. LePage, M. Woodroofe, J. Zinn, Convergence to a stable distribution via order statistics, Ann. Probab. 9 (1981) 624–632.
[35] P. Lévy, Propriétés asymptotiques des sommes de variables aléatoires indépendentes ou enchainés, J. Math. 14 (1935) 347–402.
[36] W. Liu, W.B. Wu, Asymptotics of spectral density estimates, Econometric Theory 26 (2010) 1218–1245.
[37] B.B. Mandelbrot, The variation of certain speculative price, J. Bus. 36 (1963) 394–419.
[38] B.B. Mandelbrot, The variation of other speculative prices, J. Bus. 40 (1967) 393–413.
[39] T. Mikosch, S. Resnick, G. Samorodnitsky, The maximum of the periodogram for a heavy-tailed sequence, Ann. Probab. 28 (2000) 885–908.
[40] T. Mori, On the limit distributions of lightly trimmed sums, Math. Proc. Cambridge Philos. Soc. 96 (1984) 507–516.
[41] V.V. Petrov, Limit Theorems of Probability Theory, Clarendon Press, Oxford, 1995.
[42] P.C.B. Phillips, V. Solo, Asymptotics for linear processes, Ann. Statist. 20 (1992) 971–1001.
[43] S. Resnick, P. Greenwood, A bivariate stable characterization and domains of attraction, J. Multivariate Anal. 9 (1979) 206–221.
[44] J.L. Teugels, Limit theorems on order statistics, Ann. Probab. 9 (1981) 868–880.

[45] M. Tyran-Kamińska, Functional limit theorems for linear processes in the domain of attraction of stable laws, Statist. Probab. Lett. 80 (2010) 975–981.

[46] C.S. Withers, Conditions for linear processes to be strong-mixing, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 57 (1981) 477–480.

[47] R.-M. Zhang, N.H. Chan, Maximum likelihood estimation for nearly non-stationary stable autoregressive processes, J. Time Ser. Anal. 33 (2012) 542–553.