A METRIC DISCREPANCY RESULT WITH GIVEN SPEED

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Abstract. It is known that the discrepancy $D_N\{kx\}$ of the sequence $\{kx\}$ satisfies $ND_N\{kx\} = O\big((\log N)(\log\log N)^{1+\varepsilon}\big)$ a.e. for all $\varepsilon > 0$, but not for $\varepsilon = 0$. For $n_k = \theta^k$, $\theta > 1$ we have $ND_N\{n_k x\} \le (\Sigma_\theta + \varepsilon)(2N \log\log N)^{1/2}$ a.e. for some $0 < \Sigma_\theta < \infty$ and $N \ge N_0$ if $\varepsilon > 0$, but not for $\varepsilon < 0$. In this paper we prove, extending results of Aistleitner-Larcher [6], that for any sufficiently smooth intermediate speed $\Psi(N)$ between $(\log N)(\log\log N)^{1+\varepsilon}$ and $(N \log\log N)^{1/2}$ and for any $\Sigma > 0$, there exists a sequence $\{n_k\}$ of positive integers such that $ND_N\{n_k x\} \le (\Sigma + \varepsilon)\Psi(N)$ eventually holds a.e. for $\varepsilon > 0$, but not for $\varepsilon < 0$. We also consider a similar problem on the growth of trigonometric sums.

1. Introduction

A sequence $\{x_k\}$ of real numbers is said to be uniformly distributed modulo 1 if

$$\frac{1}{N}\#\{k \le N : \langle x_k\rangle \in [\,a, b)\} \to b - a, \quad (N \to \infty),$$

for all $0 \le a < b \le 1$, where $\langle x\rangle$ denotes the fractional part $x - [\,x\,]$ of a real number x. The discrepancy $D_N\{x_k\}$, also denoted by $D_N(x_1, \dots, x_N)$, is used to measure the speed of convergence:

$$D_N\{x_k\} = \sup_{0\le a<b\le 1}\left|\frac{1}{N}\#\{k \le N : \langle x_k\rangle \in [\,a, b)\} - (b - a)\right|.$$

For arithmetic progressions $\{kx\}$ with $x \notin \mathbf{Q}$, Bohl [10], Sierpiński [24], and Weyl [26] independently proved that they are uniformly distributed modulo 1. A metric result of Khintchine [20] implies

$$ND_N\{kx\} = O\big((\log N)(\log\log N)^{1+\varepsilon}\big) \quad \text{a.e. for any } \varepsilon > 0 \qquad (1)$$

and this fails for $\varepsilon \le 0$. The discrepancy of exponentially growing sequences has also been investigated extensively. By assuming the

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Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1 \quad (k = 1, 2, \ldots), \tag{2}$$

Philipp [23] proved, using Takahashi's method [25], that

$$\frac{1}{4\sqrt{2}} \leq \varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N \log\log N}} \leq \frac{1}{\sqrt{2}}\Big(166 + \frac{664}{q^{1/2}-1}\Big) \quad \text{a.e.} \tag{3}$$

For improvements of (3), see [3] for the lower bound, and [18] for the upper bound. In case of geometric progressions, an exact law of the iterated logarithm holds: for any $\theta \notin [-1, 1]$ there exists a constant $\Sigma_\theta \geq 1/2$ with

$$\varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N \log\log N}} = \Sigma_\theta \quad \text{a.e.}$$

If $\theta^j \notin \mathbf{Q}$ for any $j \in \mathbf{N}$, then $\Sigma_\theta = \frac{1}{2}$, otherwise $\Sigma_\theta > \frac{1}{2}$. For a θ which is a power root of an integer, of a large rational number, or of a ratio of odd integers, the concrete value of Σ_θ is evaluated. See [12, 14, 15, 16, 17]. For conditions to have an exact law of the iterated logarithm in (3), see [1, 5].

Since there is a big difference between (1) and (3), it is natural to ask if for intermediate speeds $\Psi(N)$ between $(\log N)(\log\log N)^{1+\varepsilon}$ and $(N \log\log N)^{1/2}$ one can find a sequence $\{n_k\}$ of integers such that the growth speed of $D_N\{n_k x\}$ is $\Psi(N)$ in the above sense. For all $\gamma \in (0, 1/2]$, Aistleitner and Larcher [6] constructed an increasing sequence $\{n_k\}$ of integers such that $ND_N\{n_k x\} = O\big(N^\gamma\big)$ and $ND_N\{n_k x\} = \Omega\big(N^{\gamma-\varepsilon}\big)$ a.e. for all $\varepsilon > 0$. They also constructed (see [7]) a sequence $\{n_k\}$ with polynomial growth such that $ND_N\{n_k x\} = O\big((\log N)^{2+\varepsilon}\big)$ a.e. for all $\varepsilon > 0$.

The main result of the present paper is the following

Theorem 1. *Let $\{\Psi(N)\}$ be a sequence of real numbers. Assume that there exists a constant N_0 such that*

$$0 < \Psi(N) \leq \Psi(N+1) \quad \text{for all } N \geq N_0, \tag{4}$$

$$\Psi(N) \geq (\log N)(\log\log N)^{1+\varepsilon} \quad \text{for some } \varepsilon > 0 \text{ and } N \geq N_0, \tag{5}$$

$$\Psi^2(N+1) - \Psi^2(N) = o(\log\log \Psi^2(N)). \tag{6}$$

Then for any $\Sigma > 0$, there exists a sequence $\{n_k\}$ of positive integers satisfying $1 \leq n_{k+1} - n_k \leq 2$ and

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\Psi(N)} = \Sigma \quad \text{a.e.} \tag{7}$$

Note that for the function $\Psi^2(N) = N \log\log N$ we have

$$\Psi^2(N+1) - \Psi^2(N) \sim \log\log \Psi^2(N)$$

and thus condition (6) means that the jumps of $\Psi^2(N)$ are of smaller order of magnitude than those of $N \log\log N$. Naturally, this implies that $\Psi^2(N) = o(N \log\log N)$ and thus the conditions of Theorem 1 bound the function $\Psi^2(N)$ between $(\log N)(\log\log N)^{1+\varepsilon}$ and $N \log\log N$ and require a certain smoothness of growth. Typical examples are $\Psi(N) = N^\alpha (\log N)^\beta (\log\log N)^\gamma$ where the parameters α, β, γ are chosen so that the order of growth of $\Psi^2(N)$ is between the previous bounds. Note that the theorem does not cover $\Psi(N) = (N \log\log N)^{1/2}$; the existence of $\{n_k\}$ with (7) is already proved in [4] for $0 < \Sigma < \infty$, and in [2] for $\Sigma = \infty$. See also [9, 14].

As a related problem, we can ask if there exists a sequence $\{n_k\}$ such that $\sum_{k=1}^N \cos 2\pi n_k x$ grows with a given speed $\Psi(N)$. The law of the iterated logarithm by Erdős-Gál [11] states

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{N \log\log N}} \sum_{k=1}^N \cos 2\pi n_k x = 1 \quad \text{a.e.} \tag{8}$$

for $\{n_k\}$ satisfying the Hadamard gap condition (2). As we will see in Section 4, for any $D > 0$ there exists an increasing $\{n_k\}$ such that (8) holds with the norming factor replaced by $c\sqrt{N}(\log\log N)^D$. The following theorem shows that any growth speed $O(\sqrt{N}(\log\log N)^D)$ with small jumps is possible for $\sum_{k=1}^N \cos 2\pi n_k x$.

Theorem 2. *Let $\{\Psi(N)\}$ be an sequence of real numbers. Assume that there exists a constant N_0 and $D > 0$ such that (4),*

$$\Psi(N) \to \infty, \quad \text{and} \quad \Psi^2(N+1) - \Psi^2(N) \quad = o\big((\log\log \Psi^2(N))^D\big).$$

Then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that

$$\overline{\lim_{N\to\infty}} \frac{1}{\Psi(N)} \sum_{k=1}^N \cos 2\pi n_k x = 1 \quad a.e. \tag{9}$$

In conclusion, we mention a number of open problems related to our results. Let $\mathcal{G}$ denote the class of functions $\Psi(N)$, $N = 1, 2, \ldots$ such that for some increasing sequence $\{n_k\}$ relation (7) holds for some constant $0 < \Sigma < \infty$. From Theorem 1 it follows that $\mathcal{G}$ contains all smoothly increasing functions $\Psi(N)$ with speed between $(\log N)(\log\log N)^{1+\varepsilon}$ for some $\varepsilon > 0$ and $(N \log\log N)^{1/2}$. By a classical result of W. Schmidt (see e.g. Kuipers and Niedereiter [22], p. 109) for any infinite sequence $\{x_k\}$ we have $ND_N\{x_k\} \geq c \log N$ for infinitely many N with an absolute constant c and thus $\mathcal{G}$ contains no functions $\Psi(N) = o(\log N)$. Hence assumption (5) in Theorem 1 is nearly optimal; whether $\Psi(N) = (\log N)(\log\log N)^\alpha$, $0 \le \alpha \le 1$ belongs to $\mathcal{G}$ remains open. Concerning upper bounds for functions in $\mathcal{G}$, the results of Baker [8] and Berkes and Philipp [9] imply that

$$ND_N\{n_k x\} \le \text{const} \cdot N^{1/2}(\log N)^\gamma \quad \text{a.e.}$$

holds for all $\{n_k\}$ if $\gamma > 3/2$ but not if $\gamma \le 1/2$. This implies that for $\gamma > 3/2$ we have $N^{1/2}(\log N)^\gamma \notin \mathcal{G}$ and makes it plausible (but does not prove) that $(N \log N)^{1/2} \in \mathcal{G}$. If this is true, condition (6) in Theorem 1 can be replaced by

$$\Psi^2(N+1) - \Psi^2(N) = o(\log \Psi^2(N))$$

allowing all smoothly growing functions $\Psi(N) = O(N \log N)^{1/2}$, an essentially optimal result. Similar remarks hold for Theorem 2.

2. Key Proposition

We begin with proving a weaker version of Theorem 1.

Proposition 3. *For any sequence* $\{\psi(N)\}$ *satisfying*

$$\psi(0) = 0, \quad \psi(N) \le \psi(N+1), \tag{10}$$

$$(\log N)(\log\log N)^{1+\varepsilon} = o(\psi(N)) \quad \textit{for some} \quad \varepsilon > 0, \tag{11}$$

$$\psi^2(N+1) - \psi^2(N) \le \frac{1}{2}(4 \vee \log\log \psi^2(N)), \tag{12}$$

there exists a sequence $\{n_k\}$ *of positive integers satisfying* $1 \le n_{k+1} - n_k \le 2$ *and*

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\psi(N)} = \frac{\sqrt{2}}{4} \quad \textit{a.e.} \tag{13}$$

Set $G(x) = x/(4 \vee \log\log x)$, where $\log\log x$ is meant as $-\infty$ for $x \le 1$. Note that $G(x)$ is increasing. By (12), we can derive

$$G(\psi^2(N+1)) - G(\psi^2(N)) \le \frac{\psi^2(N+1) - \psi^2(N)}{4 \vee \log\log \psi^2(N)} \le \frac{1}{2}. \tag{14}$$

Let ν_i be the smallest ν satisfying $2i^3 \le G\big(\psi^2(i^3 + \nu)\big)$. Note that $\nu_0 = 0$. By (14), we have

$$G\big(\psi^2(i^3 + \nu_i)\big) = 2i^3 + e_i \quad \text{for some} \quad 0 \le e_i < 1/2. \tag{15}$$

Set $\Delta_i = \mathbf{N} \cap (2(i-1)^3, 2i^3\,]$ and $\eta_i = 2i^3 - 2(i-1)^3$.

By using (14), we have

$$\eta_i - \frac{1}{2} \le 2i^3 - 2(i-1)^3 + e_i - e_{i-1}$$
$$= G\big(\psi^2(i^3+\nu_i)\big) - G\big(\psi^2((i-1)^3+\nu_{i-1})\big) \le \frac{1}{2}\Big(\frac{1}{2}\eta_i + \nu_i - \nu_{i-1}\Big).$$

By $\eta_i \ge 2$, we have

$$\nu_i - \nu_{i-1} \ge (3/2)\eta_i - 1 \ge \eta_i \quad \text{and} \quad \nu_i \ge 2i^3. \tag{16}$$

Set $\mu_k = 2\nu_i + 2(k-2i^3)$ for $k \in \Delta_i$. By $\mu_{2i^3+1} = 2\nu_{i+1} - 2\eta_{i+1} + 2 \ge 2\nu_i + 2 > \mu_{2i^3}$, we see that $\{\mu_k\}$ is strictly increasing.

We now introduce some notation. Denote by $\mathbf{1}_{[a,b)}$ the indicator function of $[a,b)$, and put $\widetilde{\mathbf{1}}_{[a,b)}\langle x\rangle = \mathbf{1}_{[a,b)}(\langle x\rangle) - (b-a)$. Then we have

$$ND_N\{x_k\} = ND_N(x_1,\dots,x_N) = \sup_{0\le a<b\le 1}\Big|\sum_{k=1}^N \widetilde{\mathbf{1}}_{[a,b)}\langle x_k\rangle\Big|.$$

Put $S = \{2^{-l}i : l \in \mathbf{N}, i = 0,1,\dots,2^l\}$, $S^{2<} = \{(a,b) : a,b \in S, a<b\}$, $\phi_C(t) = \sqrt{Ct(1\vee \log\log t)}$, and $\sigma_{a,b} = \sqrt{(b-a)(1-(b-a))}$. Let $\{X_k\}$ be a sequence of independent random variables satisfying $P(X_k = 1) = P(X_k = -1) = 1/2$.

Lemma 4. *We have*

$$\varlimsup_{N\to\infty} \frac{1}{\phi_2(N)}\Big|\sum_{k=1}^N \widetilde{\mathbf{1}}_{[a,b)}\langle \mu_k x\rangle X_k\Big| = \sigma_{a,b} \tag{17}$$

for all $(a,b) \in S^{2<}$, *a.e., a.s.*

Proof. Since μ_k is a strictly increasing sequence of integers, by Weyl's theorem [27], $\{\mu_k x\}$ is uniformly distributed modulo 1 a.e. Hence,

$$B_N := \sum_{k=1}^N \widetilde{\mathbf{1}}^2_{[a,b)}\langle \mu_k x\rangle \sim N\int_0^1 \widetilde{\mathbf{1}}^2_{[a,b)}(y)\,dy = N\sigma^2_{a,b} \to \infty \quad \text{a.e.}$$

if $b-a \ne 0$, 1. By Kolmogorov's law of the iterated logarithm [21]

$$\varlimsup_{N\to\infty} \frac{1}{\phi_2(B_N)}\Big|\sum_{k=1}^N \widetilde{\mathbf{1}}_{[a,b)}\langle \mu_k x\rangle X_k\Big| = 1 \quad \text{a.s., a.e.,}$$

we see that (17) holds a.s., a.e. if $0 < b-a < 1$. Clearly (17) holds if $b-a = 0$, 1. Since $S^{2<}$ is countable, we see that (17) holds for all $(a,b) \in S^{2<}$, a.s., a.e. By Fubini's theorem, we have the conclusion. □

Lemma 5. *Suppose that* $l \in \mathbf{N}$ *and* $0 \le i < 2^l$, *we have*

$$\varlimsup_{N\to\infty} \frac{1}{\phi_2(N)} \sup_{0<c<2^{-l}}\Big|\sum_{k=1}^N \widetilde{\mathbf{1}}_{[2^{-l}i,2^{-l}i+c)}\langle \mu_k x\rangle X_k\Big| \le 4\cdot 2^{-l/2} \quad \textit{a.e., a.s.}$$

Proof. Denote $\mathbf{1}_{[a,b)}(\langle x\rangle)$ simply by $\mathbf{1}_{[a,b)}\langle x\rangle$. By noting

$$b_N=\sum_{k=1}^{N}\mathbf{1}_{[\,2^{-l}i,2^{-l}(i+1))}\langle\mu_k x\rangle\sim N\int_0^1\mathbf{1}_{[\,2^{-l}i,2^{-l}(i+1))}(y)\,dy=N2^{-l}\quad\text{a.e.}$$

and by following the proof of Lemma 4 of [13], we can prove

$$\varlimsup_{N\to\infty}\frac{1}{\phi_2(N)}\sup_{0<c<2^{-l}}\left|\sum_{k=1}^{N}\mathbf{1}_{[\,2^{-l}i,2^{-l}i+c)}\langle\mu_k x\rangle X_k\right|\le\sqrt{10\cdot 2^{-l}}\quad\text{a.e., a.s.}$$

Thus together with the law of the iterated logarithm

$$\varlimsup_{N\to\infty}\sup_{0<c<2^{-l}}\frac{c}{\phi_2(N)}\left|\sum_{k=1}^{N}X_k\right|=\varlimsup_{N\to\infty}\frac{2^{-l}}{\phi_2(N)}\left|\sum_{k=1}^{N}X_k\right|\le 2^{-l}\quad\text{a.s.,}$$

we have the conclusion. □

For $0\le a<b\le 1$, take l with $b-a>2^{-l}$ and take the largest i and j such that $2^{-l}i\le a<2^{-l}j\le b$. Then we have $\mathbf{1}_{[a,b)}=\mathbf{1}_{[2^{-l}i,2^{-l}j)}-\mathbf{1}_{[2^{-l}i,a)}+\mathbf{1}_{[2^{-l}j,b)}$ and $\widetilde{\mathbf{1}}_{[a,b)}=\widetilde{\mathbf{1}}_{[2^{-l}i,2^{-l}j)}-\widetilde{\mathbf{1}}_{[2^{-l}i,a)}+\widetilde{\mathbf{1}}_{[2^{-l}j,b)}$, which implies

$$\begin{aligned}
&\max_{0\le i<j\le 2^l}\varlimsup_{N\to\infty}\frac{1}{\phi_2(N)}\left|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[\,2^{-l}i,2^{-l}j)}\langle\mu_k x\rangle X_k\right|\\
&\le\varlimsup_{N\to\infty}\sup_{0<a<b\le 1}\frac{1}{\phi_2(N)}\left|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[\,a,b)}\langle\mu_k x\rangle X_k\right|\\
&\le\max_{0\le i<j\le 2^l}\varlimsup_{N\to\infty}\frac{1}{\phi_2(N)}\left|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[\,2^{-l}i,2^{-l}j)}\langle\mu_k x\rangle X_k\right|\\
&\qquad+2\max_{0\le i\le 2^l}\varlimsup_{N\to\infty}\sup_{0<a\le 2^{-l}}\frac{1}{\phi_2(N)}\left|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[\,2^{-l}i,2^{-l}i+a)}\langle\mu_k x\rangle X_k\right|.
\end{aligned}$$

By applying two lemmas above, we have

$$\frac12\le\varlimsup_{N\to\infty}\sup_{0\le a<b\le 1}\frac{1}{\phi_2(N)}\left|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[\,a,b)}\langle\mu_k x\rangle X_k\right|\le\frac12+8\cdot 2^{-l/2}\quad\text{a.e., a.s.}$$

which implies

$$\varlimsup_{N\to\infty}\sup_{0\le a<b\le 1}\frac{1}{\phi_2(N)}\left|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[\,a,b)}\langle\mu_k x\rangle X_k\right|=\frac12\quad\text{a.e., a.s.}\tag{18}$$

By the relation $ND_N\{x_k+y\}=ND_N\{x_k\}$ and (1), we have

$$\eta_i D_{\eta_i}(\mu_{2(i-1)^3+1}x,\mu_{2(i-1)^3+2}x,\dots,\mu_{2i^3}x)=\eta_i D_{\eta_i}\{2kx\}=O\big((\log\eta_i)^2\big).$$

Noting $ND_N\{\mu_k x\}\le\sum_{i=1}^{j}\eta_i D_{\eta_i}(\mu_{2(i-1)^3+1}x,\mu_{2(i-1)^3+2}x,\dots,\mu_{2i^3}x)$ for $N\in\Delta_j$, we have

$$ND_N\{\mu_k x\}=O\Bigg(\sum_{i=1}^{j}(\log\eta_i)^2\Bigg)=O\big(N^{1/3}(\log N)^2\big)=o\big(\sqrt{N}\,\big)\quad\text{a.e.}$$

by $j-1<(N/2)^{1/3}$. This together with (18) implies

$$\overline{\lim_{N\to\infty}}\sup_{0\le a<b\le 1}\frac{1}{\phi_2(N)}\Bigg|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[\,a,b)}\langle\mu_k x\rangle\frac{X_k+1}{2}\Bigg|=\frac14\quad\text{a.e., a.s.}\tag{19}$$

Note that $\{\mu_k\}$ and $\{2k-1\}$ are mutually disjoint. Let $\{\lambda_k\}$ be an arrangement in increasing order of $\{\mu_k\}\cup\{2k-1\}$. By $\mu_{2i^3}=2\nu_i$, we have $\#\{k:\mu_k\le 2\nu_i\}=2i^3$ and $\#\{k:2k-1\le 2\nu_i\}=\nu_i$, and thereby we have $\#\{k:\lambda_k\le 2\nu_i\}=2i^3+\nu_i$ and $\lambda_{2i^3+\nu_i}=2\nu_i$. We set

$$Y_k=\begin{cases}1 & \lambda_k\notin 2\mathbf{N},\\ (X_k+1)/2 & \lambda_k\in 2\mathbf{N},\end{cases}$$

$I_N=\#\{k\le N:\lambda_k\notin 2\mathbf{N}\}$, $J_N=\#\{k\le N:Y_k=1,\ \lambda_k\in 2\mathbf{N}\}$, and $H_N=\#\{k\le N:Y_k=1\}=I_N+J_N$. We have $I_{2i^3+\nu_i}=\#\{k\le 2i^3+\nu_i:\lambda_k\notin 2\mathbf{N}\}=\#\{k:2k-1\le 2\nu_i\}=\nu_i$ and $H_{2i^3+\nu_i}=J_{2i^3+\nu_i}+\nu_i$. By the law of large numbers we have $J_{2i^3+\nu_i}\sim\frac12\#\{k:\mu_k\le 2\nu_i\}=i^3$ a.s. By (14), we have

$$\big|G\big(\psi^2(H_{2i^3+\nu_i})\big)-G\big(\psi^2(i^3+\nu_i)\big)\big|\le\frac12|H_{2i^3+\nu_i}-(i^3+\nu_i)|=\frac12|J_{2i^3+\nu_i}-i^3|.$$

Dividing by $G\big(\psi^2(i^3+\nu_i)\big)=2i^2+e_i$, we have

$$\Big|\frac{G\big(\psi^2(H_{2i^3+\nu_i})\big)}{2i^3+e_i}-1\Big|\le\frac12\Big|\frac{J_{2i^3+\nu_i}}{2i^3+e_i}-\frac{i^3}{2i^3+e_i}\Big|\to 0\quad\text{a.s.}$$

Therefore we have $G\big(\psi^2(H_{2i^3+\nu_i})\big)\sim 2i^3+e_i\sim 2i^3\sim 2J_{2i^3+\nu_i}$ a.s. Since J_N and H_N are increasing, for $N\in[\,(i-1)^3+\nu_{i-1},i^3+\nu_i\,]$ we have

$$1\sim\frac{G\big(\psi^2(H_{2(i-1)^3+\nu_{i-1}})\big)}{2J_{2i^3+\nu_i}}\le\frac{G\big(\psi^2(H_N)\big)}{2J_N}\le\frac{G\big(\psi^2(H_{2i^3+\nu_i})\big)}{2J_{2(i-1)^3+\nu_{i-1}}}\sim 1,$$

and thereby,

$$2J_N\sim G\big(\psi^2(H_N)\big)\quad\text{a.s.}\tag{20}$$

By (1), we see $ND_N\{(2k-1)x\} = O\big((\log N)(\log\log N)^{1+\varepsilon/2}\big)$, which implies $ND_N\{(2k-1)x\} = o\big((\log N)(\log\log N)^{1+\varepsilon}\big)$ or

$$\lim_{N\to\infty} \sup_{0\le a<b\le 1} \frac{1}{A_N}\Bigg|\sum_{k\le N:\lambda_k\notin 2\mathbf{N}} \widetilde{\mathbf{1}}_{[a,b)}\langle\lambda_k x\rangle Y_k\Bigg| = 0 \quad \text{a.e., a.s.} \tag{21}$$

for $A_N = (\log I_N)(\log\log I_N)^{\varepsilon}$. Since $H_N \ge I_N$, it is valid for $A_N = (\log H_N)(\log\log H_N)^{\varepsilon}$. Because of (11), we see that (21) holds for $A_N = \sqrt{2}\,\psi(H_N)$.

By (19), we have

$$\overline{\lim_{N\to\infty}} \sup_{0\le a<b\le 1} \frac{1}{A_N}\Bigg|\sum_{k\le N:\lambda_k\in 2\mathbf{N}} \widetilde{\mathbf{1}}_{[a,b)}\langle\lambda_k x\rangle Y_k\Bigg| = \frac{1}{4} \quad \text{a.e., a.s.} \tag{22}$$

for $A_N = \phi_2\big(\#\{k\le N : \lambda_k \in 2\mathbf{N}\}\big)$. By $J_N \sim \frac{1}{2}\#\{k\le N : \lambda_k\in 2\mathbf{N}\}$ a.s., we see that (22) is valid for $A_N = \sqrt{2}\,\phi_2(J_N) \sim \phi_2(2J_N)$. (20) and $\phi_2^2(G(\psi^2(N))) \sim 2\psi^2(N)$ imply $\phi_2^2(J_N) \sim \phi_2^2(G(\psi^2(H_N)))/2 \sim \psi^2(H_N)$ a.s. Hence (22) holds for $A_N = \sqrt{2}\,\psi(H_N)$. Combining these, we have

$$\overline{\lim_{N\to\infty}} \sup_{0\le a<b\le 1} \frac{1}{\sqrt{2}\,\psi(H_N)}\Bigg|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}\langle\lambda_k x\rangle Y_k\Bigg| = \frac{1}{4} \quad \text{a.e., a.s.}$$

Denoting by $\{n_k\}$ the subsequence $\{\lambda_k : Y_k = 1\}$, we have (13) a.s.

3. Proof of Theorem 1

By (6), we have $\Psi^2(N) = o(N\log\log\Psi^2(N))$ and $G(\Psi^2(N)) = o(N)$. For any $C>0$, we see $G(\phi_C^2(N)) \sim CN$ and hence $G(\Psi^2(N)) \le G(\phi_C^2(N))$ or $\Psi^2(N) \le \phi_C^2(N)$ for large N. Since it holds for any $C>0$, we see that $\Psi^2(N) = o(\phi_C^2(N))$.

By (6), we can take $N_1 > N_0$ such that for all $N \ge N_1$,

$$(2\sqrt{2}\,\Sigma\Psi(N+1))^2 - (2\sqrt{2}\,\Sigma\Psi(N))^2 \le \frac{1}{2}\log\log(2\sqrt{2}\,\Sigma\Psi(N))^2. \tag{23}$$

Take $c\in(0,\frac{1}{4})$ such that $\phi_c^2(N_1) < (2\sqrt{2}\,\Sigma\Psi(N_1))^2$ holds. We have $(2\sqrt{2}\,\Sigma\Psi(N))^2 < \phi_c^2(N)$ for large $N\ge N_1$. Denote N_2 the minimum of such N. Putting

$$\psi(N) = \begin{cases} \phi_c(N) & N < N_2, \\ 2\sqrt{2}\,\Sigma\Psi(N) & N \ge N_2, \end{cases}$$

it is clear that $\psi(N)$ satisfies (10) and (11). As to the condition (12), we first prove it for $\phi_c^2(N)$.

In the case $\log\log(N+1)\ge 1$, i.e. $N\ge 15$, we see $(N+1)\big(\log\log(N+1) - \log\log N\big) \le ((N+1)/N)/\log N \le 2/\log 15 < \log\log 15 \le$

$\log\log N$ and $(N+1)\log\log(N+1) - N\log\log N < 2\log\log N$. If $c\log\log N \le 1$, then $2c\log\log N \le 2 \le \frac{1}{2}(4 \vee \log\log\phi_c^2(N))$. If $c\log\log N \ge 1$, then $2c\log\log N \le \frac{1}{2}\log\log N \le \frac{1}{2}\log\log(cN\log\log N) \le \frac{1}{2}(4 \vee \log\log\phi_c^2(N))$. Therefore, when $\log\log(N+1) \ge 1$, we have $\phi_c^2(N+1) - \phi_c^2(N) \le 2c\log\log N \le \frac{1}{2}(4 \vee \log\log\phi_c^2(N))$. When $\log\log(N+1) \le 1$, clearly we have $\phi_c^2(N+1) - \phi_c^2(N) \le c \le \frac{1}{4} \le \frac{1}{2}(4 \vee \log\log\phi_c^2(N))$.

By $\psi^2(N_2) - \psi^2(N_2-1) \le (2\sqrt{2}\,\Sigma\Psi(N_2))^2 - \phi_c^2(N_2-1) \le \phi_c^2(N_2) - \phi_c^2(N_2-1)$ together with (23), we conclude that $\psi(N)$ satisfies (12).

Hence we can apply Proposition 3 to have the conclusion.

4. Proof of Theorem 2

Take an integer $d \ge D \vee 2$ to satisfy

$$\Psi^2(N+1) - \Psi^2(N) = o\big((\log\log\Psi^2(N))^d\big). \tag{24}$$

Put $M_k = 2^{d-1}\binom{k}{d}$, $L_k = \min\{n \mid \Psi^2(n) \ge (2^{d-1}/d!)M_k(\log\log M_k)^d\}$, and $L_k^+ = L_k + M_{k+1} - M_k$.

There exists K_- such that $\max_{N\le N_0}\Psi(N) < (2^{d-1}/d!)M_k(\log\log M_k)^d$ for all $k \ge K_-$. From now on, we consider only for $k \ge K_-$, for which we have $L_k > N_0$.

By (24) and $\Psi^2(L_k-1) < (2^{d-1}/d!)M_k(\log\log M_k)^d$, we have

$$\begin{aligned}
&(2^{d-1}/d!)M_k(\log\log M_k)^d \le \Psi^2(L_k)\\
&= o\big((\log\log\Psi^2(L_k-1))^d\big) + \Psi^2(L_k-1)\\
&\le o\big((\log\log(M_k(\log\log M_k)^d)\big) + (2^{d-1}/d!)M_k(\log\log M_k)^d,
\end{aligned}$$

$\Psi^2(L_k)/(2^{d-1}/d!)M_k(\log\log M_k)^d \to 1$, $\log\log\Psi^2(L_k) - \log\log M_k \to 0$ and $\log\log\Psi^2(L_k) \sim \log\log M_k$ in turn. Combining

$$\begin{aligned}
&\Psi^2(L_{k+1}) - \Psi^2(L_k-1)\\
&\ge (2^{d-1}/d!)(M_{k+1}(\log\log M_{k+1})^d - M_k(\log\log M_k)^d)\\
&\ge (2^{d-1}/d!)(M_{k+1} - M_k)(\log\log M_{k+1})^d
\end{aligned}$$

and $\Psi^2(L_{k+1}) - \Psi^2(L_k-1) = (L_{k+1} - L_k + 1)o\big((\log\log\Psi^2(L_{k+1}))^d\big)$, we have

$$\frac{M_{k+1}-M_k}{L_{k+1}-L_k+1} \le \frac{o\big((\log\log\Psi^2(L_{k+1}))^d\big)}{(2^{d-1}/d!)(\log\log M_{k+1})^d} = o(1).$$

Hence we see that there exists a K_0 such that

$$L_{k+1} - L_k > M_{k+1} - M_k \quad \text{i.e.,} \quad L_{k+1} > L_k^+ \quad (k \ge K_0). \tag{25}$$

By (24) we have $\Psi^2(N) \le o\big(N(\log\log\Psi^2(N))^d\big)$, Thereby $\log\Psi^2(N) < \log N + d\log\log\log\Psi^2(N)$, and $\log\Psi^2(N) \le 2\log N$ or $\Psi^2(N) \le N^2$ for large N. Hence $\Psi^2(N) = o\big(N(\log\log N)^d\big)$. Hence we see $\Psi^2(M_k) = o\big(M_k(\log\log M_k)^d\big) = o\big(\Psi^2(L_k)\big)$. It implies $M_k < L_k$ for large k. Take such $k \ge K_0$ and denote by k_0. We see $M_{k_0} < L_{k_0}$.

We define an non-decreasing sequence $\{a_k\}$ of positive integers as below. Put $a_1 = \cdots = a_{k_0} = 3$, take a_{k_0+1} large enough to satisfy $a_{k_0+1} \ge a_{k_0}$ and

$$\gamma^+_{k_0+1} := \frac{1}{2}a_{k_0+1}^{k_0+1} \ge \frac{3}{2}a_{k_0}^{k_0} + (L_{k_0} - 1 - M_{k_0}) =: \gamma^-_{k_0+1}. \tag{26}$$

For $k \ge k_0$, inductively take a_{k+2} large enough to satisfy $a_{k+2} \ge a_{k+1}$ and

$$\gamma^+_{k+2} := \frac{1}{2}a_{k+2}^{k+2} \ge \frac{3}{2}a_{k+1}^{k+1} + (L_{k+1} - L_k^+) =: \gamma^-_{k+2}. \tag{27}$$

Put $\rho_j = a_j^j$. Since ρ_j satisfies the Hadamard gap condition $\rho_{j+1}/\rho_j \ge a_{j+1} \ge 3$, by the law of the iterated logarithm we have

$$\varlimsup_{N\to\infty} \frac{1}{\phi_1(N)} \sum_{j=1}^N \cos 2\pi\rho_j x = \varlimsup_{N\to\infty} \frac{1}{\phi_1(N)} \left|\sum_{j=1}^N \cos 2\pi\rho_j x\right| = 1 \quad \text{a.e.} \tag{28}$$

From this, we drive

$$\varlimsup_{N\to\infty} \frac{d!}{\phi_1(N)^d} \sum_{1\le m_1<\cdots<m_d\le N} \prod_{j=1}^d \cos 2\pi\rho_{m_j} x = 1 \quad \text{a.e.} \tag{29}$$

For a function $f(m_1,\dots,m_d)$ on $\{1,\dots,N\}^d$, define a signed measure ν on $\{1,\dots,N\}^d$ by

$$\nu(A) = \sum_{(m_1,\dots,m_d)\in A} f(m_1,\dots,m_d) \quad (A \subset \{1,\dots,N\}^d).$$

Let $J = \{(j,k) \mid 1 \le j,k \le N,\ j \ne k\}$. For $(j,k) \in J$, put $A_{(j,k)} = \{(m_1,\dots,m_d) \in \{1,\dots,N\}^d \mid m_j = m_k\}$.

Putting

$$f(m_1,\dots,m_d) = \prod_{j=1}^d \cos 2\pi\rho_{m_j} x$$

and by applying the inclusion-exclusion principle

$$\nu\Big(\{1,\dots,N\}^d \setminus \bigcup_{\boldsymbol{j}\in J} A_{\boldsymbol{j}}\Big) = \nu(\{1,\dots,N\}^d) - \sum_{\boldsymbol{j}\in J}\nu(A_{\boldsymbol{j}}) + \sum_{\boldsymbol{j}_1,\boldsymbol{j}_2\in J:\boldsymbol{j}_1\neq\boldsymbol{j}_2}\nu(A_{\boldsymbol{j}_1}\cap A_{\boldsymbol{j}_2}) - \cdots + \nu\Big(\bigcap_{\boldsymbol{j}\in J} A_{\boldsymbol{j}}\Big),$$

we see that

$$\Big|\sum_{m_1,\dots,m_d\le N: m_j\neq m_k((j,k)\in J)}\prod_{j=1}^d \cos 2\pi\rho_{m_j}x - \Big(\sum_{k=1}^N \cos 2\pi\rho_k x\Big)^d\Big|$$

can be bounded by a linear combination of

$$\Big|\prod_{j=1}^\beta\sum_{k=1}^N \cos^{\alpha_j} 2\pi\rho_k x\Big| \quad (\alpha_1+\cdots+\alpha_\beta = d,\ \max_{j=1}^\beta \alpha_j \ge 2).$$

Note that we can verify

$$\begin{aligned} 0 &\le \varlimsup_{N\to\infty}\frac{1}{\phi_1(N)^d}\Big|\prod_{j=1}^\beta\sum_{k=1}^N \cos^{\alpha_j} 2\pi\rho_k x\Big| \\ &\le \prod_{j=1}^\beta \varlimsup_{N\to\infty}\frac{1}{\phi_1(N)^{\alpha_j}}\Big|\sum_{k=1}^N \cos^{\alpha_j} 2\pi\rho_k x\Big| = 0 \quad \text{a.e.} \end{aligned}$$

because

$$\varlimsup_{N\to\infty}\frac{1}{\phi_1(N)^\alpha}\Big|\sum_{k=1}^N \cos^{\alpha} 2\pi\rho_k x\Big| \le \varlimsup_{N\to\infty}\frac{N}{\phi_1(N)^\alpha} = 0$$

holds for $\alpha\ge 2$. Hence by (28) we have

$$\begin{aligned} &\varlimsup_{N\to\infty}\frac{1}{\phi_1(N)^d}\sum_{m_1,\dots,m_d\le N: m_j\neq m_k((j,k)\in J)}\prod_{j=1}^d \cos 2\pi\rho_{m_j}x \\ &= \varlimsup_{N\to\infty}\frac{1}{\phi_1(N)^d}\Big(\sum_{k=1}^N \cos 2\pi\rho_k x\Big)^d = 1 \quad \text{a.e.} \end{aligned}$$

and thereby we see (29).

Let $\mathcal{S}_0$ be a collection of $(b_1,b_2,\dots)\in\{-1,0,1\}^{\mathbf{N}}$ such that $b_i=0$ for all large i.

Lemma 6. *The mapping $\mathcal{S}_0 \ni (b_1,b_2,\dots)\mapsto \sum_{i=1}^\infty b_i a_i^i \in \mathbf{Z}$ is injective.*

Proof. Because of $\left|\sum_{i=1}^{I-1} b_i a_i^i\right| \le \sum_{i=1}^{I-1} a_{I-1}^i < \frac{1}{2}a_I^I$, we have

$$\sum_{i=1}^{I} b_i a_i^i \in \Big(\big(b_I - \frac{1}{2}\big)a_I^I, \big(b_I + \frac{1}{2}\big)a_I^I\Big),$$

and if $b_I \neq 0$, then

$$\sum_{i=1}^{I} b_i a_i^i \in \Big(-\frac{3}{2}a_I^I, -\frac{1}{2}a_I^I\Big) \cup \Big(\frac{1}{2}a_I^I, \frac{3}{2}a_I^I\Big) =: C_I. \tag{30}$$

Take $(b_1, b_2, \dots) \in \mathcal{S}_0$ and $(b_1', b_2', \dots) \in \mathcal{S}_0$ and assume $\sum_{i=1}^{\infty} b_i a_i^i = \sum_{i=1}^{\infty} b_i' a_i^i$. By putting $I = \max\{\, i \mid b_i \neq 0\,\}$ and $I' = \max\{\, i \mid b_i' \neq 0\,\}$, then we see that $\sum_{i=1}^{\infty} b_i a_i^i \in C_I$ and $\sum_{i=1}^{\infty} b_i a_i^i \in C_{I'}$. By $\frac{3}{2}a_I^I \le \frac{1}{2}a_{I+1}^{I+1}$, we see that C_I $(I = 1,\ 2,\ \dots)$ are mutually disjoint and $\max\{\, i \mid b_i \neq 0\,\} = \max\{\, i \mid b_i' \neq 0\,\}$. Because $\Big(\big(b - \frac{1}{2}\big)a_I^I, \big(b + \frac{1}{2}\big)a_I^I\Big)$ $(b \in \mathbf{Z})$ are mutually disjoint, we see $b_I = b_I'$. Hence we have $\sum_{i=1}^{I-1} b_i a_i^i = \sum_{i=1}^{I-1} b_i' a_i^i$. In the same way, we can verify $b_i = b_i'$ for all $i < I$, and see that the mapping is injective. □

By this lemma, we see that

$$\rho_{m_d} + \varepsilon_{d-1}\rho_{m_{d-1}} + \cdots + \varepsilon_1 \rho_{m_1} \tag{31}$$

with $m_1 < m_2 < \cdots < m_d$ and $\varepsilon_1, \dots, \varepsilon_d = \pm 1$ are all distinct. Denote by $\{l_i\}$ the arrangement in increasing order of this family.

Note that M_k equals to the number of the sum of the type (31) with $m_1 < m_2 < \cdots < m_d \le k$ and $\varepsilon_1, \dots, \varepsilon_d = \pm 1$. By (30),

$$l_i \in \Big(\frac{1}{2}a_N^N, \frac{3}{2}a_N^N\Big), \quad (M_{N-1} < i \le M_N). \tag{32}$$

Clearly

$$\prod_{j=1}^{d} \cos 2\pi \rho_{m_j} x = \frac{1}{2^{d-1}} \cos 2\pi(\rho_{m_d} + \varepsilon_{d-1}\rho_{m_{d-1}} + \cdots + \varepsilon_1\rho_{m_1})x,$$

and

$$\sum_{1 \le m_1 < \cdots < m_d \le N} \prod_{j=1}^{d} \cos 2\pi\rho_{m_j} x = \frac{1}{2^{d-1}} \sum_{k=1}^{M_N} \cos 2\pi l_k x.$$

Hence by (29), we have

$$\varlimsup_{N\to\infty} \frac{d!}{2^{d-1}\phi_1(N)^d} \sum_{k=1}^{M_N} \cos 2\pi l_k x = 1 \quad \text{a.e.} \tag{33}$$

Put

$$B_N(x)=\max_{M_N+1\le Q\le M_{N+1}}\Big|\sum_{k=M_N+1}^{Q}\cos 2\pi l_k x\Big|.$$

By the Carleson-Hunt inequality [19] we have

$$\int_0^1 B_N^4(x)\,dx\le C\int_0^1\Big(\sum_{k=M_N+1}^{M_{N+1}}\cos 2\pi l_k x\Big)^4\,dx$$

where C is an absolute constant. Put

$$C_N(x)=\sum_{m_1,\dots,m_{d-1}\le N-1: m_i\ne m_j\,(i\ne j)}\prod_{j=1}^{d-1}\cos 2\pi\rho_{m_j}x.$$

By

$$\begin{aligned}\sum_{k=M_N+1}^{M_{N+1}}\cos 2\pi l_k x&=2^{d-1}\sum_{m_1<\dots<m_{d-1}<m_d=N}\prod_{j=1}^{d}\cos 2\pi\rho_{m_j}x\\&=\frac{2^{d-1}}{d!}C_N(x)\cos 2\pi Nx\end{aligned}$$

we have

$$\int_0^1 B_N^4(x)\,dx\le C\Big(\frac{2^{d-1}}{d!}\Big)^4\int_0^1 C_N^4(x)$$

As before, by the inclusion-exclusion principle, we see that $|C_N(x)|$ can be bounded from above by a linear combination of

$$\Big|\prod_{j=1}^{\beta}\sum_{k=1}^{N-1}\cos^{\alpha_j}2\pi\rho_k x\Big|\quad(\alpha_1+\cdots\alpha_\beta=d-1,\alpha_j\ge 1).$$

Put $S=\sum_{j=1}^{\beta}\alpha_j\mathbf{1}(\alpha_j>1)$ and $T=\sum_{j=1}^{\beta}\mathbf{1}(\alpha_j=1)$. $S+T=d-1$ is clear. For $\alpha\ge 2$, we bound $\big|\sum_{k=1}^{N-1}\cos^{\alpha_j}2\pi\rho_k x\big|\le N\le N^{\alpha/2}$ to have

$$\Big|\prod_{j=1}^{\beta}\sum_{k=1}^{N-1}\cos^{\alpha_j}2\pi\rho_k x\Big|\le N^{S/2}\Big|\sum_{k=1}^{N-1}\cos 2\pi\rho_k x\Big|^T.$$

By applying Theorem 8.20 of Zygmund [28], we have

$$\int_0^1\Big(\prod_{j=1}^{\beta}\sum_{k=1}^{N}\cos^{\alpha_j}2\pi\rho_k x\Big)^4\,dx=O\big(N^{2S}N^{2T}\big)=O\big(N^{2(d-1)}\big).$$

Therefore we have

$$\int_0^1 B_N^4(x)\,dx=O\big(N^{2(d-1)}\big)\quad\text{and}\quad\sum_{N=1}^{\infty}\int_0^1\Big(\frac{B_N(x)}{N^{d/2}}\Big)^4\,dx<\infty.$$

By applying the Beppo-Levi Theorem we have $B_N = o(N^{d/2})$ a.e. By noting $M_N \sim N^d 2^{d-1}/d!$ and combining with (33), we have

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{(2^{d-1}/d!)N(\log\log N)^d}} \sum_{i=1}^{N} \cos 2\pi l_i x = 1 \quad \text{a.e.} \tag{34}$$

Put

$$n_i = \begin{cases} l_i & \text{if } i \le M_{k_0}, \\ l_{M_{k_0}} + (i - M_{k_0}) & \text{if } M_{k_0} < i < L_{k_0}, \\ l_{M_k+i+1-L_k} & \text{if } L_k \le i < L_k^+, \\ n_{L_k^+-1} + (i+1-L_k^+) & \text{if } L_k^+ \le i < L_{k+1}\ (k \ge k_0), \end{cases}$$

We can verify that $\{n_k\}$ is strictly increasing. Actually by (32) and (26), we see

$$n_{L_{k_0}} = l_{M_{k_0}+1} > \gamma_{k_0+1}^+ \ge \gamma_{k_0+1}^- > l_{M_{k_0}} + (L_{k_0} - 1 - M_{k_0}) = n_{L_{k_0}-1},$$

and by (27) we see for $k \ge k_0$,

$$n_{L_{k+1}} = l_{M_{k+1}+1} > \gamma_{k+2}^+ \ge \gamma_{k+2}^- > l_{M_{k+1}} + (L_{k+1} - L_k^+) = n_{L_{k+1}-1}.$$

Put $E = [\,1, M_{k_0}\,] \cup \bigcup_{k=k_0}^{\infty} [\,L_k, L_k^+)$, $F = \mathbf{N} \setminus E$, $E_N = E \cap [\,1, N\,]$, $F_N = F \cap [\,1, N\,]$, and $\eta_N = {}^{\#}E_N$. By $\eta_{L_k} = M_k + 1$, we have $\Psi^2(L_k) \sim (2^{d-1}/d!)\eta_{L_k}(\log\log \eta_{L_k})^d$. By $\Psi^2(L_{k+1}) \sim \Psi^2(L_k)$, we have

$$\Psi^2(N) \sim (2^{d-1}/d!)\eta_N(\log\log \eta_N)^d \tag{35}$$

By (34), we see that

$$\varlimsup_{N\to\infty} \frac{1}{A_N} \sum_{i\in E_N} \cos 2\pi n_i x = 1 \quad \text{a.e.}$$

holds for $A_N = \sqrt{(2^{d-1}/d!)\eta_N \log\log \eta_N}$, and by (35) we see that it holds for $A_N = \Psi(N)$.

If $N \in [\,L_{k-1}^+, L_k)$, we have $\left|\sum_{i=L_{k-1}^+}^{N} \cos 2\pi n_i x\right| \le 2/|\sin \pi x|$. By $\Psi^2(N) \sim \eta_{L_k} \log\log \eta_{L_k} \sim (2^{d-1}/d!)k^d \log\log k$, we can see that

$$\max_{N\in[\,L_{k-1}^+, L_k)} \left| \sum_{i\in F_N} \cos 2\pi n_i x \right| \le \frac{2k}{|\sin \pi x|} = o(\Psi(N)) \quad \text{a.e.}$$

Hence we can verify (9).

References

[1] C. Aistleitner, *On the law of the iterated logarithm for the discrepancy of lacunary sequences*, Trans. Amer. Math. Soc., **362** (2010), 5967-5982.

[2] C. Aistleitner & K. Fukuyama, *On the law of the iterated logarithm for trigonometric series with bounded gaps*, Probab. Theory Related Fields, **154** (2012) 607–620.

[3] C. Aistleitner & K. Fukuyama, *Extremal discrepancy behaviour of lacunary sequences*, Monatsh. Math., **177** (2015) 167–184.

[4] C. Aistleitner & K. Fukuyama, *On the law of the iterated logarithm for trigonometric series with bounded gaps II*, J. Théor. des Nombres Bordeaux, (to appear).

[5] C. Aistleitner, K. Fukuyama & Y. Furuya, *Optimal bound for the discrepancies of lacunary sequences*, Acta Arith., **158** (2013) 229–243.

[6] C. Aistleitner & G. Larcher, *On sequences with prescribed metric discrepancy behavior*, Monatsh. Math., (to appear), DOI: 10.1007/s00605-015-0860-2

[7] C. Aistleitner & G. Larcher, *Metric results on the discrepancy of sequence* $\{a_n\alpha\}_{n\geq 1}$ *modulo one for integer sequence* $\{a_n\}$ *of polynomial growth*, Mathematika, **62** (2016) 478–491.

[8] R. Baker, *Metric number theory and the large sieve*, J. London Math. Soc. 24 (1981), 34–40.

[9] I. Berkes & W. Philipp, *The size of trigonometric and Walsh series and uniform distribution mod 1*, J. London Math. Soc. (2), **50** (1994) 454–464.

[10] P. Bohl, *Über ein in der Theorie der säkulären Störungen vorkommendes Problem*, J. Reine Angew. Math., **135** (1909), 189-283.

[11] P. Erdős & S. Gál I, *On the law of the iterated logarithm I, II*, Nederl. Akad. Wetensch. Proc. Ser. A 58 (Indag. Math. 17) (1955) 65–76, 77–84.

[12] K. Fukuyama, *The law of the iterated logarithm for discrepancies of* $\{\theta^n x\}$. Acta Math. Hungar., **118** (2008), 155-170.

[13] K. Fukuyama, *A law of the iterated logarithm for discrepancies: non-constant limsup*, Monatsh. Math., **160** (2010), 143-149.

[14] K. Fukuyama, *A central limit theorem and a metric discrepancy result for sequence with bounded gaps*, Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp, Eds. I. Berkes, R. Bradley, H. Dehling, M. Peligrad, and R. Tichy, Kendrick press, 2010, pp. 233-246.

[15] K. Fukuyama, *Metric discrepancy results for alternating geometric progressions*, Monatsh. Math., **171** (2013) 33–63.

[16] K. Fukuyama, *A metric discrepancy result for the sequence of powers of minus two*, Indag. Math. (NS), **25** (2014) 487-504.

[17] K. Fukuyama & M. Yamashita, *Metric discrepancy results for geometric progressions with large ratios*, Monatsh. Math., (to appear), DOI: 10.1007/s00605-015-0791-y.

[18] K. Fukuyama, Y. Mitsuhata, *Bounded law of the iterated logarithm for discrepancies of permutations of lacunary sequences*, Summer School on the Theory of Uniform Distribution, RIMS Kôkyûroku Bessatsu, **B29** (2012) 65–88.

[19] R. A. Hunt, *On the convergence of Fourier series, Orthogonal Expansions and their Continuous Analogues*, (Proc. Conf. Edwardsville, Ill.,1967) 235-255. Southern Illinois Univ. Press, Carbondale, Ill. 1968.

[20] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. **92**, (1924) 115-125

[21] A. Kolmogoroff, *Über das Gesetz des iterierten Logarithmus*, Math. Ann. **101**, (1929) 126-135

[22] L. Kuipers, H. Niederreiter, Uniform distribution of sequences. Wiley, 1974.

[23] W. Philipp, *Limit theorems for lacunary series and uniform distribution mod 1*, Acta Arith. **26**, (1975) 241-251

[24] W. Sierpiński, *Sur la valuer asymptotique d'une certaine somme*, Bull. Int. Acad. Polon. Sci. (Cracovie) A, **1910** (1910), 9-11.

[25] S. Takahashi, *An asymptotic property of a gap sequence*, Proc. Japan Acad., **38** (1962), 101-104.

[26] H. Weyl, *Über die Gibbssche Erscheinung und verwandte Konvergenzphänomene*, Rend. Circ. Mat. Palermo, **30** (1910) 377–407.

[27] H. Weyl, *Über die Gleichverteilung von Zahlen modulo Eins*, Math. Ann., **77** (1916), 313–352.

[28] A. Zygmund, Trigonometric series, Vol I., Cambridge University Press, 1959.

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