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ACOUSTIC BEHAVIOR OF A POROELASTIC MINDLIN PLATE

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ABSTRACT

The numerical treatment of noise insulation of solid walls has been an object of scientific research for many years. The main noise source is the bending vibration of the walls modeled by plate theory. In generally, walls consist of porous material, for instance concrete or bricks. To model the effects of the porous structure more realistic it is advantageous to formulate a plate theory based on poroelastic constitutive equations.

A theory for poroelastic Kirchhoff plates was presented by Beskos and Theodorakoupoulos. It is well known that for the approximation of the behavior of thin plates, where shear deformation and rotatory inertia can be neglected, this theory is very useful. However, if the structure under consideration becomes thicker, e.g., a wall, the refined theory of Reissner/Mindlin is required.

The most common used theory of dynamic poroelasticity was developed by Biot formulated either using the solid and fluid displacements as unknowns or, more physically motivated, using the solid displacements and the pore pressure as unknowns. In this work the latter is used.

After establishing the poroelastic plate theory by incorporating the classical kinematic assumptions of the Mindlin plate theory a variational principle for the poroelastic plate is developed leading, finally, to a Finite Element formulation. Based on this formulation, the flexural vibration of a Mindlin plate for varied material data is presented.

Keywords: Biot's theory, poroelasticity Mindlin plate theory.

INTRODUCTION

The vibrations of porous structures are important in problems of sound absorption and in aeronautical industry. As one application the sound transmission through porous walls can be mentioned. Porous walls are mainly modeled by the elastic Mindlin plate theory (Mindlin 1951) using an elastic bending stiffness achieved by an homogenization process. A more detailed model is achieved by using a poroelastic theory, e.g., Biot's theory, in the framework of Mindlin's plate theory. This theory of porous materials containing a viscous fluid was presented by Biot in 1941. In the following years, Biot extended his theory to the anisotropic case (Biot 1955) and also to poroviscoelasticity (Biot 1956a). The dynamic extension was done by Biot in 1956b.

The combination of this refined constitutive assumption and a plate theory is published by Theodorakopoulos and Beskos (1994). They used the Kirchhoff theory assuming thin plates and neglected any in-plane motion to obtain purely bending vibrations. The opposite approach

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where any bending is neglected and only the in-plane motion of a poroelastic thin disc is considered may be found in Cederbaum et al. (2000). A simplified theory of Theodorakopoulos and Beskos (1994) is used by Leclaire and Horoshenkov (2001) assuming incompressible constituents, i.e., a rigid elastic skeleton and incompressible fluid.

Here, a general poroelastic theory of Biot is used and a Mindlin plate is considered to model moderately thick plates, e.g., walls. Therefore, first, Biot's theory is recalled followed by the application of Mindlin assumptions of the plate. Subsequently, a variational formulation is presented to prepare a Finite Element formulation for the proposed poroelastic Mindlin plate. The example of a hinged poroelastic square plate closes the presentation.

POROELASTIC GOVERNING EQUATIONS

Following Biot's approach to model the behavior of porous media, an elastic skeleton with a statistical distribution of interconnected pores is considered (Biot 1955). This porosity is denoted by ϕ. Contrary to these pores the sealed pores will be considered as part of the solid. Full saturation is assumed in the following.

One possible representation of poroelastic constitutive equation is obtained using the total stress $\sigma_{ij} = \sigma^s_{ij} + \sigma^f \delta_{ij}$ and the pore pressure p as independent variables (Biot 1941). Introducing Biot's effective stress coefficient α and the solid displacement u_i the constitutive equation reads

$$\sigma_{ij} = G\left(u_{i,j} + u_{j,i}\right) + \frac{2G\nu}{1-2\nu} u_{k,k}\delta_{ij} - \alpha\delta_{ij} p \tag{1}$$

with the shear modulus of the solid frame G and Poisson's ratio ν. In this equation, and in the following, Latin indices take the values 1,2,3, where summation convention is implied over repeated indices and the spatial derivative with respect to the coordinate x_i is denoted by $()_{,i}$. Further, in (1) a linear strain-displacement relation is taken into account assuming small deformation gradients. Additional to the total stress σ_{ij}, as a second constitutive equation the variation of fluid volume per unit reference volume ζ is introduced

$$\zeta = \alpha u_{k,k} + \frac{\phi^2}{R} p \tag{2}$$

with the material constant R. This variation of fluid ζ is defined by the mass balance over a reference volume, i.e., by the continuity equation

$$\frac{\partial\zeta}{\partial t} + q_{i,i} = 0 \tag{3}$$

with the specific flux $q_i = \phi \frac{\partial v_i}{\partial t}$ and the relative fluid to solid displacement v_i.

Additional to the fluid balance (3), the balance of momentum for the bulk material must be fulfilled. This dynamic equilibrium is given by

$$\sigma_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} + \phi\rho_f \frac{\partial^2 v_i}{\partial t^2} \ , \tag{4}$$

with the bulk body force per unit volume F_i and the bulk density $\rho = \rho_s(1-\phi) + \phi\rho_f$. The density of the solid and the fluid is denoted by ρ_s and ρ_f, respectively.

Next, the fluid transport in the interstitial space expressed by the specific flux q_i is modeled with a generalized Darcy's law

$$q_i = -\kappa\left(p_{,i} + \rho_f \frac{\partial^2 u_i}{\partial t^2} + \frac{\rho_a + \phi\rho_f}{\phi}\frac{\partial^2 v_i}{\partial t^2}\right) \ , \tag{5}$$

where κ denotes the permeability. In equation (5), an additional density the apparent mass density ρ_a is introduced by Biot (1956b) to describe the interaction between fluid and skeleton. It can be written as $\rho_a = C\phi\rho_f$ where C is a factor depending on the geometry of the pores and the frequency of excitation (for details see, Biot (1956b)).

Aiming at the equation of motion, the above balance laws and constitutive equations have to be combined to obtain a set of coupled differential equations for the unknowns solid displacements u_i and the pore pressure p. First, Darcy's law (5) is rearranged to obtain v_i. Since v_i is given as second time derivative in (5), this is only possible in frequency domain. After transformation to frequency domain, i.e., assuming a steady state caused by harmonic excitation with frequency ω, the relative fluid to solid displacement is ($\hat{()}$ denotes the transformed function)

$$\hat{v}_i = \underbrace{\frac{\kappa\rho_f\phi^2 \mathrm{i}\omega}{\phi^2 + \mathrm{i}\omega\kappa\left(\rho_a + \phi\rho_f\right)}}_{\beta} \; \frac{1}{\omega^2\phi\rho_f}\left(\hat{p}_{,i} - \omega^2\rho_f\hat{u}_i\right) \; . \tag{6}$$

In equation (6), the abbreviation β is defined for further usage. Now, the final set of differential equations for the displacement $\hat{u}_i$ and the pore pressure $\hat{p}$ is obtained by inserting the constitutive equation (2) in the dynamic equilibrium (4) and continuity equation (3) with $\hat{v}_i$ from equation (6). This leads to

$$\hat{\sigma}_{ij,j} + \beta\hat{p}_{,i} + \omega^2\left(\rho - \beta\rho_f\right)\hat{u}_i = -\hat{F}_i \tag{7}$$

$$\frac{\beta}{\mathrm{i}\omega\rho_f}\hat{p}_{,ii} - \frac{\phi^2\mathrm{i}\omega}{R}\hat{p} - \left(\alpha - \beta\right)\mathrm{i}\omega\hat{u}_{i,i} = 0 \; . \tag{8}$$

This set of equations describes the behavior of a poroelastic continuum completely. Aiming at the acoustic behavior of a plate consisting of poroelastic material a frequency dependent formulation is sufficient, i.e., no inverse transformation of (7) and (8) is necessary. Note, this would only be possible for $\kappa \to \infty$.

POROELASTIC MINDLIN PLATE THEORY

After formulating the 3-d governing equations the usual plate assumptions for a Mindlin plate will be inserted. To ensure that there are no in-plane motions the in-plane forces $F_1 = F_2 = 0$ and the in-plane flux $q_1 = q_2 = 0$ will be neglected.

So, following Mindlin's plate theory, the displacements are replaced by

$$\hat{u}_\alpha\left(x_1, x_2, x_3\right) = x_3\psi_\alpha\left(x_1, x_2\right) \qquad \hat{u}_3\left(x_1, x_2, x_3\right) = w\left(x_1, x_2\right) \tag{9}$$

with the frequency dependent rotations ψ_α and the frequency dependent deflection w. Above and in the following, Greek indices α, β take the values 1,2 where also summation convention is implied over repeated indices. Further, it is assumed that the in-plane normal stresses are much larger than the out-of-plane normal stress

$$\sigma_{33} \ll \sigma_{11}, \sigma_{22} \quad \longrightarrow \quad \sigma_{33} \approx 0 \; , \tag{10}$$

which is used to eliminate the derivative of the out-of-plane displacement $\hat{u}_3$ with respect to the x_3-direction

$$\sigma_{33} = 0 \qquad \longrightarrow \quad \hat{u}_{3,3} = \frac{\alpha\left(1 - 2\nu\right)}{2G\left(\left(1 - \nu\right)\right.}\hat{p} - \frac{\nu}{1 - \nu}\, x_3\psi_{\alpha,\alpha} \; . \tag{11}$$

Concerning the shear stresses in the x_3-direction the same assumption as for an elastic plate

$$\hat{\sigma}_{\alpha 3} = \frac{E}{2\,(1+\nu)}\,(\psi_\alpha + w_{,\alpha}) \quad \alpha = 1,2 \tag{12}$$

is used, i.e., these shear stresses are assumed to be constant over the thickness. This assumption is not touched by the poroelastic constitutive law because the pore fluid influences only the volumetric part of the stress tensor and not the deviatoric part. Consequently, as for an elastic plate, also here, the approximation introduced by the assumption of constant shear stress $\sigma_{\alpha 3}$ can be improved by using the shear corrected thickness h_s.

The next steps are similar to the deduction of an elastic plate. First, the above given assumptions are inserted in the out-of-plane direction $i = 3$ of equation (7) and an integration over the thickness from $-h/2$ to $h/2$ is performed yielding the governing equation for the deflection

$$G\,(\psi_{\alpha,\alpha} + w_{,\alpha\alpha})\,h_s + \beta\,(p_o - p_u) + \int_{-\frac{h}{2}}^{\frac{h}{2}} F_3\,\mathrm{d}x_3 + \omega^2\,(\rho - \beta\rho_f)\,hw = 0\,. \tag{13}$$

The pore pressure on the upper side of the plate $p_o = \hat{p}(h/2)$ and on the lower side of the plate $p_u = \hat{p}(-h/2)$ are due to the partial integration performed on $\int_{-h/2}^{+h/2} \hat{p}_{,33}\,\mathrm{d}x_3$ during the above explained integration over the thickness.

Next, the equations for the rotations are achieved by using the in-plane directions $i = 1,2$ of equation (7). Before integrating them over the thickness they are multiplicated by x_3. After eliminating $\hat{u}_{3,3}$ with (11) the equations for the rotations read

$$\begin{aligned} \frac{h^3}{12} G\psi_{\alpha,\beta\beta} + \frac{h^3}{12} G\frac{1+\nu}{1-\nu}\psi_{\beta,\alpha\beta} - \left(\alpha\frac{1-2\nu}{1-\nu} - \beta\right) Q_{,\alpha} + h_s G\,(\psi_\alpha + w_{,\alpha}) \\ + \omega^2\,(\rho - \beta\rho_f)\,\frac{h^3}{12} = 0\,. \end{aligned} \tag{14}$$

In equation (14), a new abbreviation $Q = \int_{-h/2}^{+h/2} x_3 \hat{p}(x_1, x_2, x_3)\,dx_3$ is inserted.

The last step is to perform the same operations also on the equation for the pore pressure (8), i.e., multiplication with x_3, integration over the thickness, and eliminating of $\hat{u}_{3,3}$. This yields the equation for the integrated pore pressure Q

$$\begin{aligned} \frac{\beta}{\mathrm{i}\omega\rho_f} Q_{,\alpha\alpha} - \mathrm{i}\omega\left[\frac{\phi^2}{R} + (\alpha-\beta)\,\frac{\alpha\,(1-2\nu)}{2G\,(1-\nu)}\right] Q - \mathrm{i}\omega\,(\alpha-\beta)\,\frac{1-2\nu}{1-\nu}\frac{h^3}{12}\psi_{\alpha,\alpha} \\ + \frac{\beta}{\mathrm{i}\omega\rho_f}\left[\frac{h}{2}\,(q_o - q_u) + (p_u - p_o)\right] = 0 \end{aligned} \tag{15}$$

where the normal derivative of the pore pressure on the upper and lower surface of the plate is abbreviated by $q_o = \hat{p}_{,3}\,(h/2)$ and $q_u = \hat{p}_{,3}\,(-h/2)$. With the equations (13), (14), and (15) sufficient equations for the unknowns deflection w, rotation ψ_α, and integrated pore pressure Q are given. Looking for the pore pressure p explicitely an assumption on its distribution over the thickness must be made to calculate the integral Q.

VARIATIONAL PRINCIPLE AND FE FORMULATION

To formulate, finally, a Finite Element Method for the proposed poroelastic plate, first, the principle of virtual work must be given. To deduce this principle a weighted residual statement is used where the integration over the domain is reduced to an integration over the surface of the plate A because the integration over the thickness is still performed in the above given differential equations.

So, for the deflection this is achieved by multiplication of the deflection governing equation (13) with the variation of the deflection δw and integrating it over A. The next part of the principle of virtual work is obtained by equating the rotation governing equation (14) with the variation of the rotation $\delta\psi_\alpha$ and integration over A. This procedure is quite usual in plate theory. Here, additional the third equation for the integrated pore pressure (15) has to be given. Consequently, as weighting function the variation of the integrated pore pressure δQ is chosen. Finally, all three described parts are summed up and a partial integration yields the principle of virtual work. The explicit expression are skipped here due to lack of space.

Choosing ansatz function in this principle of virtual work results in a Finite Element formulation. Usual in poroelastic FEM the pore pressure is approximated with an ansatz order less than the deflection and rotation (Lewis and Schrefler 1998). Here, for first testing, the same ansatz order is applied on all unknowns pore pressure, deflection, and rotation. This yields the usual FEM matrix representation

$$\left(\mathbf{K}-\omega^2\mathbf{M}\right)\begin{pmatrix}\mathbf{w}\\ \psi_1\\ \psi_2\\ \mathbf{p}\end{pmatrix}=f\,. \tag{16}$$

As usual in poroelastic u-p FEM formulations the stiffness matrix and the mass matrix are un-symmetric and frequency dependent. The representation in (16) with the ω^2 in front of the mass matrix is only used to have the well known structure of classical FEM.

EXAMPLE

To test the method, the example shown in figure 1 is implemented in a FEM-program. A uniformly distrubuted dynamic load is assigned to a hinged plate with the dimensions $0.2\,\mathrm{m}\times 0.2\,\mathrm{m}\times 0.01\,\mathrm{m}$. The deflection at point P ($0.125\,\mathrm{m}\times 0.1\,\mathrm{m}$) is observed. The plate is discretized using 256 linear elements and the material data are those of a rock (Berea sandstone), see table 1. There, water as well as air are used as interstitial fluid. The data of the water filled rock are taken from literature (Cheng et al. 1991). Those for the air filled rock are calculated from the other data by changing the fluid density and fluid compressibility but not the permeability.

TABLE 1. Material data for Berea sandstone (rock)

	$K\,[\frac{N}{m^2}]$	$G\,[\frac{N}{m^2}]$	$\rho\,[\frac{kg}{m^3}]$	ϕ	$R\,[\frac{N}{m^2}]$	$\rho_f\,[\frac{kg}{m^3}]$	α	$\kappa\,[\frac{m^4}{Ns}]$
rock with H_2O	$8\cdot 10^9$	$6\cdot 10^9$	2458	0.19	$4.7\cdot 10^8$	1000	0.778	$1.9\cdot 10^{-10}$
rock with air	$8\cdot 10^9$	$6\cdot 10^9$	2268	0.19	$4.7\cdot 10^8$	1.3	0.778	$1.9\cdot 10^{-10}$

In figure 2, the deflection of the rock plate is plotted versus the frequency for different values of porosity. A shift of the three visible eigenfrequencies can be recognized. For smaller values of the porosity ($\phi = 0.1$) than that of the measured data ($\phi = 0.19$) the eigenfrequencies become smaller and for larger values of the porosity($\phi = 0.3, \phi = 0.4$) they are shifted to higher frequencies.

In figure 3, a rock plate with water filled pores and a rock plate with air filled pores are com-

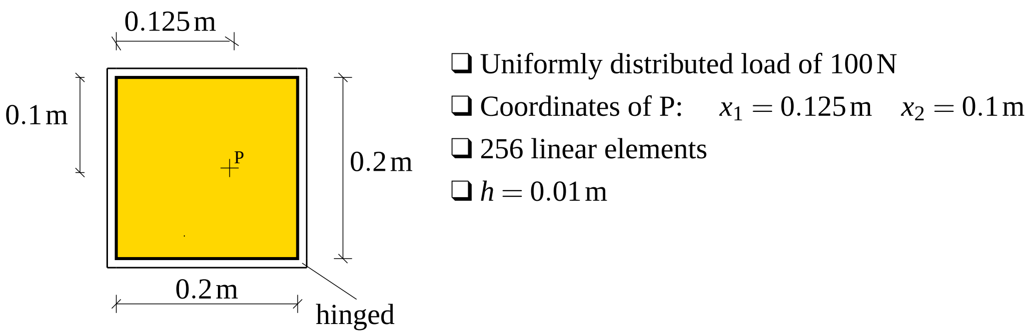

FIG. 1. Test example: Hinged rock plate

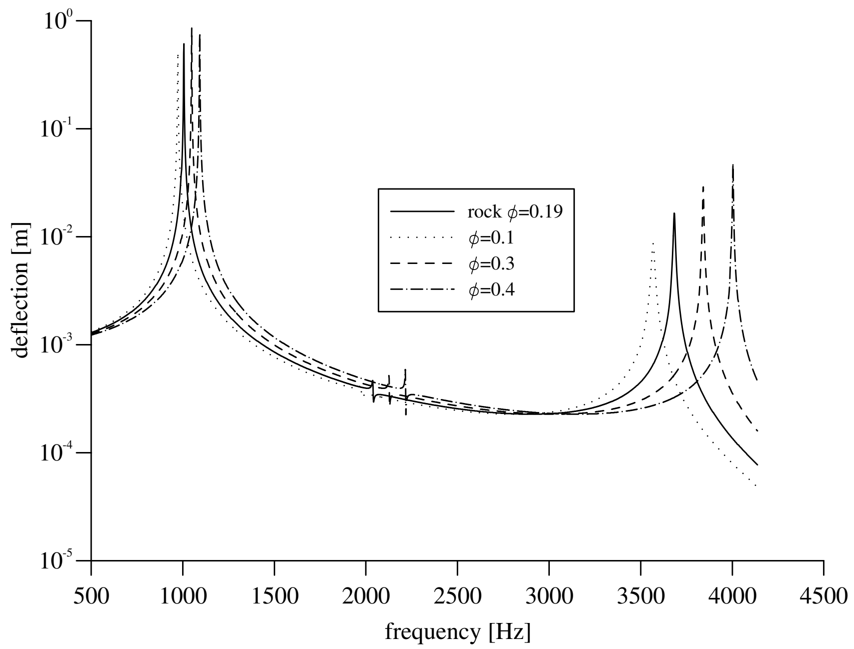

FIG. 2. Deflection at point P: Variation of the porosity ϕ of a rock plate

pared. Again the deflection is plotted versus the frequency. A dependence of the eigenfrequencies from the interstitial fluid can be recognized. For the air filled plate the eigenfrequencies are shifted to higher frequencies.

CONCLUSIONS

In the presented paper, the governing equations for a poroelastic Mindlin plate are given. Essentially, the same steps as for an elastic plate are performed where additionally in the equation of the pore pressure also the kinematic assumption of the Mindlin theory are included. As long as no assumption on the spatial behavior of the pore pressure in the thickness direction is made, only the integrated pore pressure over the thickness can be calculated. Finally, a prin-

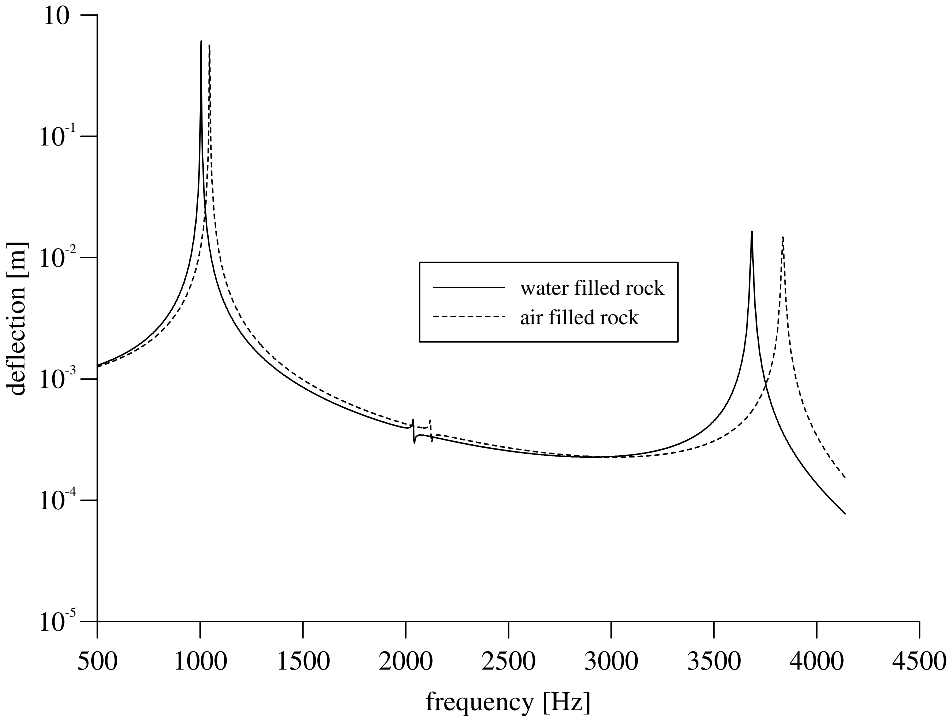

FIG. 3. Deflection at point P: Comparison of rock plates with water and air filled pores

ciple of virtual work is formulated as basis for a Finite Element formulation. The method is tested using as example a hinged rock plate. The change of porosity as well as of the interstitial fluid mainly causes a shift in the eigenfrequencies.

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